Isotonic Estimation: The Asymptotic Distribution

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**Isotonic Regression.** Consider an isotonic regression model

\[ y_k = \phi\left(\frac{k}{n}\right) + \epsilon_k, \quad k = 1, \ldots, n, \]

where \( \phi \) is non-decreasing, and \( \epsilon_1, \ldots, \epsilon_n \) are i.i.d. errors with mean 0, finite variance \( \sigma^2 \), and a finite moments generating function near 0. Let \( G_n^\# \) denote the normalized cumulative sum diagram; thus, \( G_n^\# \) is a continuous piecewise linear function for which

\[ G_n^\#\left(\frac{k}{n}\right) = \frac{1}{n} \sum_{j=1}^{k} y_j \]

for \( j = 1, \ldots, n \). Also, let

\[ \Phi(t) = \int_0^t \phi(s)ds \]

and let \( \Phi^\# \) be a continuous piecewise linear function for which

\[ \Phi_n^\#\left(\frac{k}{n}\right) = \frac{1}{n} \sum_{j=1}^{k} \phi\left(\frac{j}{n}\right). \]

Then, under modest conditions (in particular, if \( \phi \) has a bounded derivative),

\[ \sup_t |\Phi_n^\#(t) - \Phi(t)| = O\left(\frac{1}{n}\right); \]

and then

\[ G_n^\#(t) = \Phi(t) + \frac{\sigma}{\sqrt{n}} B_n(t) + R_n(t), \]

where \( B_n \) is a Brownian motion and

\[ \sup_t |R_n(t)| = O\left(\frac{\log(n)}{n}\right) \text{ w.p.1.} \]
Now, fix a $0 < t < 1$ and suppose that $\phi$ has a positive continuous derivative $\phi'$ on some neighborhood of $t$. Then

$$G^\#(t + n^{-\frac{1}{3}}s) - G^\#(t) - \phi(t)n^{-\frac{1}{3}}s = [\Phi(t + n^{-\frac{1}{3}}s) - \Phi(t) - \phi(t)n^{-\frac{1}{3}}s] + \frac{\sigma}{\sqrt{n}}[\mathcal{B}_n(t + n^{-\frac{1}{3}}s) - \mathcal{B}_n(t)] + [R_n(t + n^{-\frac{1}{3}}s) - R_n(t)],$$

Observe that $W_n(s) = n^{\frac{2}{3}}[\mathcal{B}_n(t + n^{-\frac{1}{3}}s) - \mathcal{B}_n(t)]$ is a two-sided Brownian motion and let

$$Z_n(s) = n^{\frac{2}{3}}[G^\#_n(t + n^{-\frac{1}{3}}s) - G^\#_n(t) - \phi(t)n^{-\frac{1}{3}}s]$$

and

$$Z^o_n = \sigma W_n(s) + \frac{1}{2} \phi'(t)s^2,$$

and observe that the distribution of $Z^o_n$ does not depend on $n$. Then

$$Z_n(s) = Z^o_n(s) + \gamma_n(s) + R^t_n(s)$$

where

$$R^t_n(s) = n^{\frac{2}{3}}[R_n(t + n^{-\frac{1}{3}}s) - R_n(t)] = O \left[ \frac{\log(n)}{n^{1/3}} \right] \text{ w.p.1}$$

and

$$\gamma_n(s) = n^{\frac{2}{3}}[\Phi(t + n^{-\frac{1}{3}}s) - \Phi(t) - \phi(t)n^{-\frac{1}{3}}s - \frac{1}{2} \phi'(t)n^{-\frac{2}{3}}s^2].$$

Here $\gamma_n(s) \to 0$ as $n \to \infty$ uniform on $|s| \leq c$ for any $0 < c < \infty$ and, therefore, for some sequence $c = c_n \to \infty$. So

$$\sup_{|s| \leq c_n} |Z_n(s) - Z^o_n(s)| \to^p 0.$$

Next, let $\tilde{G}_n$ be the greatest convex minorant of $G^\#_n$, and recall that the least square estimator of $\phi$ is $\hat{\phi}_n(t) = \tilde{G}'_{n,t}(t)$. Recall too that if $f$ is bounded and $h$ is linear, then $(f + h) = \tilde{f} + h$. So,

$$\tilde{G}_n(t + n^{-\frac{1}{3}}s) - \tilde{G}_n(t) - \phi(t)n^{-\frac{1}{3}}s = [\tilde{G}^\#_n(t + n^{-\frac{1}{3}}s) - \tilde{G}^\#_n(t) - \phi(t)n^{-\frac{1}{3}}s].$$

Let $\tilde{Z}_n$ and $\tilde{Z}^o_n$ denote the greatest convex minorants of the restrictions of $Z_n$ and $Z^o_n$ to $|s| \leq c_n$. Then with probability approaching one

$$n^{\frac{2}{3}}[\tilde{G}_n(t + n^{-\frac{1}{3}}s) - \tilde{G}^\#_n(t) - \phi(t)n^{-\frac{1}{3}}s] = \tilde{Z}_n(s)$$
Finally, let $W(s), -\infty < s < \infty$, be a standard two-sided Brownian motion, $Z_{a,b}(s) = aW(s) + bs^2$, $Z(s) = Z_{1,1}$, and let $\tilde{Z}_{a,b}$ be the greatest convex minorant (on $\mathbb{R}$) of $Z_{a,b}$. Then, using the fact that the distribution of $Z_{a,b}$ is the same as the distribution of $Z_{\sigma,\frac{1}{2}\phi'(t)}$ the restrictions of $\tilde{Z}_n$ and $\tilde{Z}_{a,b}$ to any finite interval converge to those of (the restriction) of $\tilde{Z}_{\sigma,\frac{1}{2}\phi'(t)}$. In particular,

$$\tilde{Z}_n|_{[-1,1]} \Rightarrow \tilde{Z}_{\sigma,\frac{1}{2}\phi'(t)}|_{[-1,1]}.$$ 

As a corollary

$$n^{\frac{1}{2}}[\hat{\phi}_n(t) - \phi(t)] = n^{\frac{1}{2}}[\hat{G}'_{n,x}(t) - \phi(t)] = \tilde{Z}'_{n,x}(0)$$

with probability approaching one. So,

$$n^{\frac{1}{2}}[\hat{\phi}_n(t) - \phi(t)] \Rightarrow \tilde{Z}'_{\sigma,\frac{1}{2}\phi'(t)}(0),$$

by the continuous mapping principle.

**On the Limiting Distribution.** Let $Z_{\ast}(t) = Z_{a,b}(ct), -\infty < t < \infty$. Then, by rescaling, $Z_{\ast}(s) = aW(cs) + b(cs)^2$ has the same distribution as $a\sqrt{c}W(s) + bc^2s^2$. So, letting $c = (a/b)^{2/3}$, $Z_{a,b} \Leftrightarrow a^{\frac{4}{3}}b^{-\frac{1}{3}}Z$. Letting $a = \sigma$ and $b = \phi'(t)/2$, it follows that

$$\tilde{Z}_n|_{[-1,1]} \Rightarrow a^{\frac{4}{3}}b^{-\frac{1}{3}}Z|_{[-1,1]},$$

$$\tilde{Z}'_{n}(0) \Rightarrow \frac{1}{c}a^{\frac{4}{3}}b^{-\frac{1}{3}}\tilde{Z}'(0) = (a^2b)^{\frac{1}{3}}\tilde{Z}'(0),$$

and

$$n^{\frac{1}{2}}[\hat{\phi}_n(t) - \phi(t)] \Rightarrow \kappa\tilde{Z}'(0),$$

where

$$\kappa = \left[\frac{1}{2}\sigma^2\phi'(t)\right]^\frac{1}{3}.$$ 

The distribution of $\tilde{Z}'(0)$ first appeared in [1] in a different context. It is a tabulated in [2] which can be consulted for a (slightly biased collection of) further references. The distribution has shorter tails than the standard normal.
References
