Ch 2 \[ P_{68} - P_{60} \]

24. \( i = 2 \), \((1,1)\) \( p(i=2) = \frac{1}{36} \)
\( i = 3 \), \((1,2), (2,1)\) \( p(i=3) = \frac{2}{36} = \frac{1}{18} \)
\( i = 4 \), \((1,3), (2,2), (3,1)\) \( p(i=4) = \frac{3}{36} = \frac{1}{12} \)
\( i = 5 \), \((1,4), (2,3), (3,2), (4,1)\) \( p(i=5) = \frac{4}{36} = \frac{1}{9} \)
\( i = 6 \), \((1,5), (2,4), (3,3), (4,2), (5,1)\) \( p(i=6) = \frac{5}{36} \)
\( i = 7 \), \((1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\) \( p(i=7) = \frac{6}{36} = \frac{1}{6} \)
\( i = 8 \), \((2,6), (3,5), (4,4), (5,3), (6,2)\) \( p(i=8) = \frac{5}{36} \)
\( i = 9 \), \((3,6), (4,5), (5,4), (6,3)\) \( p(i=9) = \frac{4}{36} = \frac{1}{9} \)
\( i = 10 \), \((4,6), (5,5), (6,4)\) \( p(i=10) = \frac{3}{36} = \frac{1}{12} \)
\( i = 11 \), \((5,6), (6,5)\) \( p(i=11) = \frac{2}{36} = \frac{1}{18} \)
\( i = 12 \), \((6,6)\) \( p(i=12) = \frac{1}{36} \) total points 6

25. 5 can be rolled in 4 ways \((1,4), (2,3), (3,2), (4,1)\)
7 can be rolled in 6 ways \((1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\)

\( E_n \) = events that 5 occurs on the \( n \)th roll and no 5 or 7 occurs on the first \( n-1 \) rolls
\[ p(E_n) = \left(\frac{36 - 6 - 4}{36}\right)^{n-1} \cdot \frac{4}{36} = \left(\frac{26}{36}\right)^{n-1} \cdot \frac{4}{36} \]
\[ \sum_{n=1}^{\infty} p(E_n) = \frac{4}{36} \cdot \sum_{n=1}^{\infty} \left(\frac{26}{36}\right)^{n-1} = \frac{4}{36} \cdot \frac{1}{1 - \frac{26}{36}} = \frac{8}{5} \text{ total points 8} \]

41. \( p(6 \text{ comes up at least once}) = 1 - p(6 \text{ doesn't come up in the 4 rolls}) \)
\[ = 1 - \frac{5^4}{6^4} \text{ total points 4} \]

407. every stranger has their birthday on different months.
\[ p = \frac{12!}{12^{12}} \text{ total points 4} \]
51. First choose the m balls from the 5 balls so that will fall in the first compartment is \( \binom{m}{n} \) choices, then the other \( n-m \) balls are randomly distributed in \( N-1 \) compartments is \( \binom{n}{N-1}^{n-m} \) choices.

so \( P(\text{m balls fall in the first compartment}) = \frac{\binom{m}{n} \binom{n}{N-1}^{n-m}}{n^m} \).

53. Let \( A_i \) be the event that couple \( i \) sit next to each other.

Then \( 1 - P \left( \bigcup_{i=1}^{4} A_i \right) \) is the desired probability.

\[
P \left( \bigcup_{i=1}^{4} A_i \right) = \sum_{i=1}^{4} P(A_i) - \sum_{i < j} P(A_i A_j) + \sum_{i < j < k} P(A_i A_j A_k) - P(A_1 A_2 A_3 A_4)
\]

\[
P(A_i) = \frac{2 \cdot 7!}{8!}
\]

\[
P(A_i A_j) = \frac{2 \cdot 6!}{8!}
\]

\[
P(A_i A_j A_k) = \frac{2 \cdot 5!}{8!}
\]

so the desired probability \( = 1 - 4 \cdot \frac{2 \cdot 7!}{8!} + 6 \cdot \frac{2 \cdot 6!}{8!} - 4 \cdot \frac{2 \cdot 5!}{8!} + \frac{2 \cdot 4!}{8!} \)

\[
P = \frac{9}{8} \quad \text{total points 8}
\]

55. Define \( F_i = E_i \cup F \), \( F_2 = E_2 \cap E_1^c \), \( \ldots \), \( F_i = E_i \cap \bigcap_{j=1}^{i-1} E_j^c \).

11. \( 1 = P(E \cup F) = P(E) + P(F) - P(EF) \).

so \( P(E) + P(F) - 1 \leq P(EF) \).

let \( P(E) = .9 \), \( P(F) = .8 \), \( P(EF) = .9 + .8 - 1 = .7 \) \text{ total points 6}

12. because \( P(EF) + P(EF^c) = P(E) \), \( P(EF) + P(EF^c) = P(F) \).

so \( P(EE^c \cup EF^c) = P(EE^c) + P(EF^c) \) : \( EF^c \) and \( EF^c \) are disjoint sets.

\[
= (P(E) - P(EF)) + (P(F) - P(EF))
\]

\[
= P(E) + P(F) - 2P(EF)
\]

\text{total points 6}
16. $P(E_1 E_2 \ldots E_n) \geq P(E_1 E_2 \ldots E_{n-1}) + P(E_n) - 1$ by Bonferroni’s Inequality

$\geq P(E_1 \ldots E_{n-2}) + P(E_{n-1}) + P(E_n) - 2$.

by induction hypothesis

$\geq \sum_{i=1}^{n} P(E_i) - (n-1)$. total points 6

19. A total of $k$ balls will be withdrawn if there are $r-1$ red balls in the first $k-1$ withdrawals and the $k$th withdrawal is a red ball.

then the probability that the first $k-1$ withdrawals has $r-1$ red balls

is $\frac{(n-r)(m-r)}{(n+m-1)}$ and the probability that the $k$th withdrawal is a red ball

is $\frac{n-r+1}{n+m-k+1}$ so $P = \frac{(n-r)(m-r)(n-r+1)}{(n+m-k+1)(n+m-k+1)}$ total points 8

20. The experiment would be flipping a coin until you toss a head.

Like $H, TH, TTH, TTTH, \ldots$ so the probability of the $n$th time you

toss a head is $\frac{1}{2^n}$, this experiment whose sample space consists of a

countably infinite number of points and all points have positive probability

which is $\frac{1}{2^n}$, $n=1, 2, \ldots$ But not all points can be equally likely,

Or the probabilities would not be able to add up to 1.

P64 ~ P65 total points 6

6. Let $R$ be the events that both balls are red. Let $B$ be the events

that they are both black. $P(R) = \frac{3 \cdot 4}{6 \cdot 10} \quad P(B) = \frac{3 \cdot 6}{6 \cdot 10}$

$P(R) + P(B) = \frac{1}{2}$. total points 6
9. Let \( S = \bigcup_{i=1}^{n} A_i \), and consider the experiment of randomly choosing an experiment of \( S \). Then \( P(A) = N(A) \frac{1}{N(S)} \). From proposition 4.4,

\[
P(A_1 \cup A_2 \cup \ldots \cup A_n) = \sum_{i=1}^{n} P(A_i) - \sum_{i<j}^{n} P(A_i, A_j) + \ldots + (-1)^{r+1} \sum_{i_1 < i_2 < \ldots < i_r} P(A_{i_1}, A_{i_2}, \ldots, A_{i_r}) + \ldots + (-1)^{n+1} P(A_1 A_2 \ldots A_n)
\]

\[
\frac{N(A_1 \cup A_2 \cup \ldots \cup A_n)}{N(S)} = \sum_{i=1}^{n} \frac{N(A_i)}{N(S)} - \sum_{i<j}^{n} \frac{N(A_i, A_j)}{N(S)} + \ldots + (-1)^{n+1} \frac{N(A_1 A_2 \ldots A_n)}{N(S)}
\]

Let \( S = A_1 U A_2 U \ldots U A_n \).

10. Let \( B_i = A_1 \), \( B_i = A_i \left( \bigcup_{j=1}^{i-1} A_j \right)^c \), \( i > 1 \). Then \( P(\bigcup_{i=1}^{n} A_i) = P(\bigcup_{i=1}^{n} B_i) \) and \( B_i \) are disjoint events, so \( P(\bigcup_{i=1}^{n} B_i) = \sum_{i=1}^{n} P(B_i) \leq \sum_{i=1}^{n} P(A_i) \)

so \( P(\bigcup_{i=1}^{n} A_i) \leq \sum_{i=1}^{n} P(A_i) \)

15. \( P(\bigcup_{i=1}^{n} A_i) = 1 - P\left(\left(\bigcap_{i=1}^{n} A_i\right)^c\right) = 1 - P\left(\bigcup_{i=1}^{n} A_i^c\right) \geq 1 - \sum_{i=1}^{n} P(A_i^c) = 1 \)

result from 14. problem.

total points: 6

whole points: 100