On the association of sum- and max-stable processes

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A B S T R A C T

We address the notion of association of sum- and max-stable processes from the perspective of linear and max-linear isometries. We establish the appealing result that these two classes of isometries can be identified on a proper space (the extended positive ratio space). This yields a natural way to associate to any max-stable process a sum-stable process. By using this association, we establish connections between structural and classification results for sum- and max-stable processes.

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1. Introduction


The deep connection between heavy-tailed sum- and max-stable processes has long been suspected. As observed, for example, in Davis and Resnick (1989) the moving maxima and the moving averages are statistically indistinguishable in the extremes. Also, the maxima of independent copies of a sum-stable process (appropriately rescaled) converge in distribution to a max-stable process and the two processes have very similar spectral representations (see e.g. Theorem 5.1 in Stoev and Taqqu (2006)). In Stoev (2008), the ergodic properties of max-stable processes were characterized by borrowing ideas and drawing parallels to existing work in the sum-stable domain. Recently Kabluchko (2009) codified the notion of association between sum- and max-stable processes. This was done by essentially identifying the spectral measures of the finite-dimensional distributions of the two processes. In this paper we provide another, equivalent, definition of association between sum- and max-stable processes, which is based on relating their spectral functions. The combination of two perspectives provides a more clear picture about the fundamental connections between the two classes of processes.

In the rest of this section, we briefly introduce the main results of this paper. We will only consider infinite variance symmetric $\alpha$-stable (SaS, $\alpha \in (0, 2)$) sum-stable processes and $\alpha$-Fréchet max–stable processes. Recall that an infinite variance SaS variable $X$ has characteristic function $\phi_X(t) = E \exp(-itX) = \exp(-\sigma^\alpha |t|^\alpha)$, $\forall t \in \mathbb{R}$, where $\alpha \in (0, 2)$. On the other hand, $Y$ has an $\alpha$-Fréchet distribution if $F_Y(y) = P(Y \leq y) = \exp(-\sigma^\alpha y^{-\alpha})$, $\forall y \in (0, \infty)$, where now $\alpha$ is in $(0, \infty)$. The $\sigma$’s in both cases are positive parameters referred to as scale coefficients. Observe that the form of the characteristic function $\phi_X$ of an SaS variable is preserved under multiplication, which reflects the sum-stability of this class of distributions. Similarly, the form of the distribution function $F_Y$ is preserved under multiplication, which corresponds to stability with respect to taking independent maxima.

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More similarities can be observed on the level of processes. Recall that \(X = \{X_t\}_{t \in T}\) is an \(\alpha\)\text{-S} stochastic process if all its finite linear combinations \(\sum_{i=1}^{n} a_i X_{t_i}, a_i \in \mathbb{R}, t_i \in T\), are \(\alpha\)\text{-S}. These processes have convenient integral (or spectral) representations:

\[
\{X_t\}_{t \in T} \overset{d}{=} \left\{ \int_S f_t(s) M_{\alpha,+}(ds) \right\}_{t \in T}.
\]

(1.1)

Here \(\{f_t\}_{t \in T} \subset L^\alpha(S, \mu)\), \(\int\cdot\) stands for the stable integral and \(M_{\alpha,+}\) is an \(\alpha\)\text{-S} random measure on measure space \((S, \mu)\) with control measure \(\mu\) (see Chapters 3 and 13 in Samorodnitsky and Taqqu, 1994).

On the other hand, \(Y = \{Y_t\}_{t \in T}\) is an \(\alpha\)\text{-Fréchet} max-stable process if all its positive max-linear combinations \(\bigvee_{i=1}^{n} a_i Y_{t_i}, a_i \geq 0, t_i \in T\) are \(\alpha\)\text{-Fréchet}. Such processes have extremal integral representations of the form

\[
\{Y_t\}_{t \in T} \overset{d}{=} \left\{ \int_S f_t(s) M_{\alpha,\vee}(ds) \right\}_{t \in T}.
\]

(1.2)

Here \(\{f_t\}_{t \in T} \subset L_\alpha^+(S, \mu) := \{ f \in L^\alpha(S, \mu) : f \geq 0 \}, \int\cdot\) stands for the extremal integral and \(M_{\alpha,\vee}\) is an \(\alpha\)\text{-Fréchet} random sup-measure with control measure \(\mu\) (see e.g. Stoev and Taqqu, 2006 and de Haan, 1984 for an alternative treatment). The \(f_t\)'s in (1.1) and (1.2) are called the spectral functions of the sum- or max-stable processes, respectively. In this paper, we will assume \((S, \mu)\) to be a standard Lebesgue space (see Appendix A in Pipiras and Taqqu, 2004) and let \(T\) denote an arbitrary index set, which may be sometimes equipped with a measure \(\lambda\). Two common settings are \((S, \mu) = ([0, 1], \text{Leb}), T = \mathbb{Z}\) with \(\lambda\) being the counting measure and \(T = \mathbb{R}\) with \(\lambda\) being the Lebesgue measure. A brief review on stable and extremal integrals is given in Table 1 in Section 2.

The representation (1.1) implies that

\[
\mathbb{E} \exp \left\{ -i \sum_{j=1}^{n} a_j X_{t_j} \right\} = \exp \left\{ -\int_S \left| \sum_{j=1}^{n} a_j f_{t_j}(s) \right|^\alpha \mu(ds) \right\}, \quad a_j \in \mathbb{R}, t_j \in T,
\]

(1.3)

which determines the finite-dimensional distributions (f.d.d.) of the \(\alpha\)\text{-S} process \(\{X_t\}_{t \in T}\). The f.d.d. of the \(\alpha\)\text{-Fréchet} process \(\{Y_t\}_{t \in T}\) in (1.2), on the other hand, are expressed as:

\[
\mathbb{P}(Y_{t_1} \leq a_1, \ldots, Y_{t_n} \leq a_n) = \exp \left\{ -\int_S \left( \bigvee_{j=1}^{n} f_{t_j}(s)/a_j \right)^\alpha \mu(ds) \right\}, \quad a_j \geq 0, t_j \in T.
\]

(1.4)

Note that the r.h.s. of (1.3) and (1.4) are similar. Indeed, they both involve exponentials of either linear (\(\sum\)) or max-linear (\(\bigvee\)) combinations of spectral functions. The characterizations (1.3) and (1.4) and their close connections play an important role throughout this paper. In particular, the key result behind our notion of association is the following theorem.

**Theorem 1.1.** Consider two arbitrary collections of functions \(f^{(1)}_1, \ldots, f^{(n)}_n \in L^\alpha(S_i, \mu_i), i = 1, 2, 0 < \alpha < 2\). Then,

\[
\left\| \sum_{j=1}^{n} a_j f^{(1)}_j \right\|_{L^\alpha(S_1, \mu_1)} = \left\| \sum_{j=1}^{n} a_j f^{(2)}_j \right\|_{L^\alpha(S_2, \mu_2)}, \quad \text{for all } a_j \in \mathbb{R},
\]

(1.5)

if and only if

\[
\left\| \bigvee_{j=1}^{n} a_j f^{(1)}_j \right\|_{L_v^\alpha(S_1, \mu_1)} = \left\| \bigvee_{j=1}^{n} a_j f^{(2)}_j \right\|_{L_v^\alpha(S_2, \mu_2)}, \quad \text{for all } a_j \geq 0.
\]

(1.6)

The proof of this result is given in Section 3. We now briefly discuss its role. It is well known that the spectral representations in (1.1) and (1.2) are not unique. Let \(\{g^{(i)}_t\}_{t \in T} \subset L^\alpha(S_i, \mu_i), i = 1, 2\) be two spectral representations of an \(\alpha\)\text{-S} process \(\{X_t\}_{t \in T}\). Relation (1.3) shows that (1.5) holds with \(g^{(i)}_t\) replaced by \(f^{(i)}_t\), \(i = 1, 2\). Relation (1.4) and Theorem 1.1 then imply that \(\{g^{(i)}_t\}_{t \in T}, i = 1, 2\) are two spectral representations of the same \(\alpha\)\text{-Fréchet} process \(\{Y_t\}_{t \in T}\). Conversely, two equivalent spectral representations of an \(\alpha\)\text{-Fréchet} process \(\{Y_t\}_{t \in T}\) yield also equivalent representations of an \(\alpha\)\text{-S} process \(\{X_t\}_{t \in T}\). Therefore, we define the association as follows:

**Definition 1.1 (Associated \(\alpha\)\text{-S} and \(\alpha\)\text{-Fréchet Processes).** We say that an \(\alpha\)\text{-S} process \(\{X_t\}_{t \in T}\) and an \(\alpha\)\text{-Fréchet} process \(\{Y_t\}_{t \in T}\) are associated, if there exist \(\{f_t\}_{t \in T} \subset L^\alpha(S, \mu)\) such that:

\[
\{X_t\}_{t \in T} \overset{d}{=} \left\{ \int_S f_t dM_{\alpha,+} \right\}_{t \in T} \quad \text{and} \quad \{Y_t\}_{t \in T} \overset{d}{=} \left\{ \int_S f_t dM_{\alpha,\vee} \right\}_{t \in T}.
\]

In this case, we say \(\{X_t\}_{t \in T}\) and \(\{Y_t\}_{t \in T}\) are associated by \(\{f_t\}_{t \in T}\).
Table 1  
Basic properties of stable and extremal stochastic integrals.

<table>
<thead>
<tr>
<th></th>
<th>(S_{\alpha}S, \alpha \in (0, 2), f \in L^\alpha(S, \mu))</th>
<th>(\alpha)-Fréchet, (\alpha \in (0, \infty), f \in L^\alpha(S, \mu))</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Stochastic integral:</strong></td>
<td>(Z = \int_{\mathbb{R}} f(s) M_{\alpha}(ds)) is (S_{\alpha}S)</td>
<td>(Z = \int_{\mathbb{R}} f(s) M_{\alpha}(ds)) is (\alpha)-Fréchet</td>
</tr>
<tr>
<td><strong>Scale coefficient:</strong></td>
<td>(|Z|<em>\alpha = (\int</em>{\mathbb{R}}</td>
<td>f</td>
</tr>
<tr>
<td>Independently scattered:</td>
<td>(\int_{\mathbb{R}} f_1 dM_{\alpha,+} + \int_{\mathbb{R}} f_2 dM_{\alpha,+} \Rightarrow f_3 = 0)</td>
<td>(\int_{\mathbb{R}} f_1 dM_{\alpha,+} + \int_{\mathbb{R}} f_2 dM_{\alpha,+} \Rightarrow f_3 = 0)</td>
</tr>
<tr>
<td>Linearity: (\forall \alpha, f \in L^\alpha(S, \mu))</td>
<td>(\int_{\mathbb{R}} (a_1 f_1 + a_2 f_2) dM_{\alpha,+} = a_1 \int_{\mathbb{R}} f_1 dM_{\alpha,+} + a_2 \int_{\mathbb{R}} f_2 dM_{\alpha,+})</td>
<td>(\int_{\mathbb{R}} (a_1 f_1 + a_2 f_2) dM_{\alpha,+} = a_1 \int_{\mathbb{R}} f_1 dM_{\alpha,+} + a_2 \int_{\mathbb{R}} f_2 dM_{\alpha,+})</td>
</tr>
</tbody>
</table>

**Theorem 1.1** ensures the consistency of this definition (see **Theorem 4.1**) and it also shows that our notion of association is not merely formal. For example, stationary or self-similar max-stable processes are associated with stationary or self-similar sum–stable ones, respectively (see **Corollary 4.1**).

We will also see, however, that there are \(S_{\alpha}S\) processes that cannot be associated to any \(\alpha\)-Fréchet processes (see **Theorem 4.2**). In particular, we provide a practical characterization of the max-associable \(S_{\alpha}S\) processes \(\{X_t\}_{t \in \mathbb{T}}\) with stationary increments characterized by dissipative flow, indexed by \(T = \mathbb{R}\) or \(T = \mathbb{Z}\) (see **Proposition 4.1**).

The paper is organized as follows. In Section 2, some preliminaries are provided. In Section 3, we prove **Theorem 1.1**. In Section 4, we establish the association of \(S_{\alpha}S\) and \(\alpha\)-Fréchet processes and give examples of both max-associable and non-max-associable \(S_{\alpha}S\) processes. In Section 5, we show how the association can serve as a tool to translate available structural results for \(S_{\alpha}S\) processes to \(\alpha\)-Fréchet processes, and vice versa. We conclude with a comparison between our approach and that of Kabluchko (2009) in Section 6.

2. **Preliminaries**

The basic properties of the stable and extremal stochastic integrals involved in Representations (1.1) and (1.2) are summarized in Table 1. For more details, see e.g. Samorodnitsky and Taqqu (1994) and Stoev and Taqqu (2006).

In the rest of this section, we draw a connection between the linear isometries and max-linear isometries, which play important roles in defining two representations of a given \(S_{\alpha}S\) process and, respectively. The notion of a linear isometry is well known. To define a max-linear isometry, we use the notation \(\bigwedge F \subset L^\alpha(S, \mu)\) is a max-linear space if for all \(n \in \mathbb{N}\), \(f_1, \ldots, f_n \in F\) and if \(F\) is closed w.r.t. the metric \(\rho_{\mu, \alpha}(f, g) = \int_{\mathbb{R}} |f^\alpha - g^\alpha| d\mu\).

**Definition 2.1 (Max-linear Isometry).** Let \(\alpha > 0\) and consider two measures \((S_1, \mu_1)\) and \((S_2, \mu_2)\) with positive and \(\sigma\)-finite measures \(\mu_1\) and \(\mu_2\). Let \(F_1 \subset L^\alpha(S_1, \mu_1)\) be a max-linear space. A mapping \(U : F_1 \rightarrow L^\alpha(S_2, \mu_2)\), is said to be a max-linear isometry, if: (i) for all \(f_1, f_2 \in F_1\) and \(a_1, a_2 \geq 0\), \(U(a_1 f_1 \vee a_2 f_2) = a_1 U(f_1) \vee a_2 U(f_2)\), \(\mu_2\)-a.e. and (ii) for all \(f \in F_1\), \(\|Uf\|_\alpha^{\mu_2(S_2, \mu_2)} = \|f\|_\alpha^{\mu_1(S_1, \mu_1)}\).

A linear (max-linear resp.) isometry may be defined only on a small linear (max-linear resp.) subspace of \(L^\alpha(S, \mu)\). Hence, this linear (max-linear resp.) isometry can be extended uniquely to the extended ratio space (extended positive ratio space resp.), which will turn out to be closed w.r.t. both linear and max-linear combinations.

**Definition 2.2 (Extended Ratio Spaces).** Let \(F\) be a collection of functions in \(L^\alpha(S, \mu)\).

(i) The ratio \(\sigma\)-field of \(F\), written \(\rho(F) := \sigma\{f_1/f_2, f_1, f_2 \in F\}\), is defined as the \(\sigma\)-field generated by ratio of functions in \(F\), with the conventions \(\pm 1/0 = \pm \infty\) and \(0/0 = 0\).

(ii) The extended ratio space of \(F\), written \(\mathcal{R}_e(F)\), is defined as:

\[
\mathcal{R}_e(F) := \{rf : rf \in L^\alpha(S, \mu), r \sim \rho(F), f \in F\}.
\]  

Similarly, we define extended positive ratio space:

\[
\mathcal{R}_{e,+}(F) := \{rf : rf \in L^\alpha_+(S, \mu), r \sim \rho(F), r \geq 0, f \in F\}.
\]  

The following result is due to Hardin (1981) and Wang and Stoev (2009).

**Theorem 2.1.** Let \(F\) be a linear (max-linear resp.) subspace of \(L^\alpha(S_1, \mu_1)\) with \(0 < \alpha < 2\) (\(L^\alpha(S_1, \mu_1)\) with \(0 < \alpha < \infty\) resp.). If \(U\) is a linear (max-linear resp.) isometry from \(F\) to \(U(F)\), then \(U\) can be uniquely extended to a linear (max-linear resp.) isometry \(\overline{U} : \mathcal{R}_e(F) \rightarrow \mathcal{R}_e(U(F))\) (\(\overline{U} : \mathcal{R}_{e,+}(F) \rightarrow \mathcal{R}_{e,+}(U(F))\) resp.), with the form

\[
\overline{U}(rf) = T(r)U(f),
\]  

for all \(rf \in \mathcal{R}_e(F)\) in (2.1) (\(rf \in \mathcal{R}_{e,+}(F)\) as in (2.2) resp.). Here \(T\) is the mapping from \(L^\alpha(S_1, \rho(U(F)), \mu_1)\) to \(L^\alpha(S_1, \rho(U(F)), \mu_2)\), induced by a regular set isomorphism \(T\) from \(\rho(F)\) to \(\rho(U(F))\).
For the precise definition of a regular set isomorphism $T$ and the induced mapping $\overline{T}$, see Lamperti (1958), Hardin (1981) or Wang and Stoev (2009). The following remark provides some intuition. Part (iii) is especially important since it shows that the two types of isometries can be identified.

Remark 2.1. (i) $\overline{U}$ is well defined in the sense that for any $r f_i \in \mathcal{R}_e(\mathcal{F})$, $i = 1, 2$ in (2.1), if $r f_1 = r f_2$, $\mu$-a.e., then $\overline{U}(r f_1) = \overline{U}(r f_2)$, $\mu_2$-a.e. Similar result holds for $r f_i \in \mathcal{R}_{e+}(\mathcal{F})$ as in (2.2).

(ii) $\overline{T}$ maps any two almost disjoint sets to almost disjoint sets. See Lamperti (1958).

(iii) The mapping $\overline{T}$ is both linear and max-linear, i.e., for $a, b \geq 0$,

$$\overline{T}(af + bg) = a\overline{T}f + b\overline{T}g \quad \text{and} \quad \overline{T}(af \lor bg) = a\overline{T}f \lor b\overline{T}g.$$  \hspace{1cm} (2.4)

This follows from the definition $\overline{T}1_A = 1_{T(A)}$ for measurable $A \subset S_1$ and the construction of $\overline{T}$ via simple functions. It is via $\overline{T}$ that the linearity and max-linearity are identified.

To make good use of (iii) in Remark 2.1, we introduce the notion of positive-linearity. We say a linear isometry $U$ is positive-linear, if $U$ maps all nonnegative functions to nonnegative functions. Accordingly, we say that $\mathcal{F} \subset L^n(S, \mu)$ is a positive-linear space, if it is closed w.r.t. the metric $\rho_{\mu, \alpha}$ and all positive-linear combinations, i.e., for all $n \in \mathbb{N}, f_i \in \mathcal{F}, a_i \geq 0$, we have $g := \sum_{i=1}^{n} a_i f_i \in \mathcal{F}$. Note that the metric $(f, g) \mapsto \|f - g\|_{L^n(S, \mu)}$ restricted to $L^n(S, \mu)$ generates the same topology as the metric $\rho_{\mu, \alpha}$. Clearly, Theorem 2.1 holds if $\mathcal{F}$ is a positive-linear (instead of a linear) subspace of $L^n(S, \mu)$. In this case, $\overline{U}$ is also positive-linear. We conclude this section with the following refinement of statement (iii) in Remark 2.1.

Proposition 2.1. Let $U$ be as in Theorem 2.1. If $\mathcal{F}$ is a positive-linear subspace of $L^n(S_1, \mu_1)$, then the linear isometry $\overline{U}$ in (2.3) is also a max-linear isometry from $\mathcal{R}_{e+}(\mathcal{F})$ to $\mathcal{R}_{e+}(U(\mathcal{F}))$. If $\mathcal{F}$ is a max-linear subspace of $L^n(S_1, \mu_1)$, then the max-linear isometry $\overline{U}$ in (2.3) is also a positive-linear isometry from $\mathcal{R}_{e}(\mathcal{F})$ to $\mathcal{R}_{e}(U(\mathcal{F}))$.

Proof. Suppose $\mathcal{F}$ is max-linear and $\overline{U}$ is a max-linear isometry. We show $\overline{U}$ is also positive-linear. First, if $\overline{U}$ in (2.3) is max-linear, then the mapping $\overline{T}$ from $L^n(S_1, \rho(\mathcal{F}), \mu_1)$ to $L^n(S_2, \rho(U(\mathcal{F})), \mu_2)$ is both max-linear and linear, by Remark 2.1(iii). Moreover, it is easily seen that $\mathcal{P}$ is positive-linear. Now, for $r f_1, r f_2 \in \mathcal{R}_{e+}(\mathcal{F})$ as in (2.2), we have

$$\overline{U}(r f_1 \lor r f_2) = \overline{U}(U f_1 \lor U f_2) = a_1 \overline{U}(r f_1) + a_2 \overline{U}(r f_2).$$

That is, $\overline{U}$ is positive-linear. The proof of the other case is similar, except that we need the existence of full support function $f$ in $\mathcal{F}$, guaranteed by Lemma 3.2 in Hardin (1981).

3. Identification of max-linear and positive-linear isometries

In this section we prove Theorem 1.1. It will be used to relate SoS and $\alpha$-Fréchet processes in the next section. To do so, we need to introduce a subspace of $L^n(S, \mu)$ which is closed w.r.t. the max-linear and positive-linear combinations. For any $F \subset L^n(S, \mu)$, let

$$\mathcal{F}_+ \coloneqq \text{span}_+\{F\} \quad \text{and} \quad \mathcal{F}_\lor \coloneqq \lor\text{-span}\{F\}$$

(3.1)

denote the smallest positive-linear and max-linear subspace of $L^n(S, \mu)$ containing the collection of functions $F$, respectively. We call them the max-linear and positive-linear spaces generated by $F$, respectively. (We also write $\mathcal{F} := \text{span}\{F\}$ as the smallest linear subspace of $L^n(S, \mu)$ containing $F$.) In general, we have $\mathcal{F}_+ \neq \mathcal{F}_\lor$. This means both $\mathcal{F}_+$ and $\mathcal{F}_\lor$ are too small to be closed w.r.t. both "$\lor$" and "$\lor$" operators. However, these two subspaces generate the same extended positive ratio space, on which the two types of isometries are identical. The following fact is proved in the Appendix.

Proposition 3.1. Suppose $F \subset L^n(S, \mu)$. Then $\mathcal{R}_{e+}(\mathcal{F}_+) = \mathcal{R}_{e+}(\mathcal{F}_\lor)$.

Proof of Theorem 1.1. Let $F^{(i)} \coloneqq \{f^{(i)}_1, \ldots, f^{(i)}_j\} \subset L^n(S_i, \mu_i)$. We prove the "if" part. Suppose Relation (1.6) holds and we will show (1.5). Relation (1.6) implies that there exists a unique max-linear isometry $U$ from $\mathcal{F}_\lor^{(1)}$ onto $\mathcal{F}_\lor^{(2)}$, such that $U f^{(i)}_j = f^{(i)}_j$, $1 \leq j \leq n$. Thus, Theorem 2.1 implies that the mapping

$$U : \mathcal{R}_{e+}(\mathcal{F}_\lor^{(1)}) \to \mathcal{R}_{e+}(U(\mathcal{F}_\lor^{(1)}))$$

with form (2.3) is a max-linear isometry. By Proposition 2.1, we have that $\overline{U}$ is also a positive-linear isometry. By Proposition 3.1, $\overline{U}$ is a positive-linear isometry defined on $\mathcal{R}_{e+}(\mathcal{F}_+^{(1)})$, which implies (1.5). The proof of the 'if' part is similar. □
To conclude this section, we will address the following question: for \( f_1^{(1)}, \ldots, f_n^{(1)} \in L^a(S_1, \mu_1) \), do there always exist nonnegative \( f_1^{(2)}, \ldots, f_n^{(2)} \in L^a_+(S_2, \mu_2) \) such that Relation (1.5) holds for any \( a_j \in \mathbb{R}^2 \)? The answer is ‘No’. As a consequence, in the next section we will see that there are \( SoS \) processes, which cannot be associated to any \( \alpha \)-Fréchet process.

**Proposition 3.2.** Consider \( f_j^{(1)} \in L^a(S_1, \mu_1), 1 \leq j \leq n \). Then, there exist some \( f_j^{(2)} \in L^a_+(S_2, \mu_2), 1 \leq j \leq n \) such that (1.5) holds, if and only if

\[
 f_j^{(1)}(s)f_j^{(1)}(s) \geq 0, \mu_1\text{-a.e. for all } 1 \leq i, j \leq n. 
\]  

(3.2)

When (3.2) is true, one can take \( f_j^{(2)}(s) := |f_j^{(1)}(s)|, 1 \leq i \leq n \) and \( (S_2, \mu_2) \equiv (S_1, \mu_1) \) for (1.5) to hold.

The proof is given in the Appendix. We will call (3.2) the associable condition.

4. Association of max- and sum-stable processes

In this section, by essentially applying Theorem 1.1, we associate an \( SoS \) process to every \( \alpha \)-Fréchet process by Definition 1.1. The associated processes will be shown to have similar properties. However, we will also see that not all the \( SoS \) processes can be associated to \( \alpha \)-Fréchet processes. We conclude with several examples.

**Remark 4.1.** In Definition 1.1, the associated \( SoS \) and \( \alpha \)-Fréchet processes have the same \( \alpha \in (0, 2) \). It is easy to see that, for any \( \alpha \)-Fréchet process \( \{Y_t\}_{t \in T} \) with spectral functions \( \{f_t^{(1)}\}_{t \in T} \) is \( \alpha/\beta \)-Fréchet with spectral functions \( \{f_t^{(2)}\}_{t \in T} \), for all \( 0 < \alpha, \beta < \infty \). This transformation shows that the parameter \( \alpha \) plays essentially no role in characterizing the dependence structure of the \( \alpha \)-Fréchet process. Given an \( SoS \) process with nonnegative spectral functions, one could associate it to the 1-Fréchet process with spectral functions \( \{f_t^{(1)}\}_{t \in T} \). This leads to no loss of generality. Here, we chose to pair up the two \( \alpha \)’s for technical convenience.

The following result, a simple application of Theorem 1.1, shows the consistency of Definition 1.1, i.e., the notion of association is independent of the choice of the spectral functions.

**Theorem 4.1.** Suppose an \( S \alpha S \) process \( \{X_t\}_{t \in T} \) and an \( \alpha \)-Fréchet process \( \{Y_t\}_{t \in T} \) are associated by \( \{f_t^{(1)}\}_{t \in T} \subseteq L^a(S_1, \mu_1) \). Then, \( \{f_t^{(2)}\}_{t \in T} \subseteq L^a_+(S_2, \mu_2) \) is a spectral representation of \( \{X_t\}_{t \in T} \) if and only if it is a spectral representation of \( \{Y_t\}_{t \in T} \). Namely,

\[
\left\{ \int_{S_1} f_t^{(1)}(s) d\mu_{a, +} \right\}_{t \in T} \overset{d}{=} \left\{ \int_{S_2} f_t^{(2)}(s) d\mu_{a, +} \right\}_{t \in T} \iff \left\{ \int_{S_1} f_t^{(1)}(s) d\mu_{a, -} \right\}_{t \in T} \overset{d}{=} \left\{ \int_{S_2} f_t^{(2)}(s) d\mu_{a, -} \right\}_{t \in T},
\]  

(4.1)

where \( \mu_{a, +} \) and \( \mu_{a, -} \) are \( S \alpha S \) random measures and \( \alpha \)-Fréchet random sup-measures, respectively, on \( S \), with control measure \( \mu_\alpha \), \( i = 1, 2 \).

As an immediate consequence, stationarity and self-similarity are preserved under association. Here we assume \( T = \mathbb{R}^d \) or \( \mathbb{Z}^d \).

**Corollary 4.1.** Suppose an \( S \alpha S \) process \( \{X_t\}_{t \in T} \) and an \( \alpha \)-Fréchet process \( \{Y_t\}_{t \in T} \) are associated. Then,

(i) \( \{X_t\}_{t \in T} \) is stationary if and only if \( \{Y_t\}_{t \in T} \) is stationary.

(ii) \( \{X_t\}_{t \in T} \) is self-similar with exponent \( H \), if and only if \( \{Y_t\}_{t \in T} \) is self-similar with exponent \( H \).

**Proof.** Suppose \( \{X_t\}_{t \in T} \) and \( \{Y_t\}_{t \in T} \) are associated by \( \{f_t^{(1)}\}_{t \in T} \subseteq L^a_+(S, \mu) \). (i) For any \( h \in T \), letting \( g_t = f_{t+h}, \forall t \in T \), by stationarity of \( \{X_t\}_{t \in T} \), we obtain \( \{g_t\}_{t \in T} \) as another spectral representation. Namely, \( \{\int_S g_t d\mu_{a, +}\}_{t \in T} \overset{d}{=} \{\int_S f_t d\mu_{a, +}\}_{t \in T} \). By Theorem 4.1, the previous statement is equivalent to \( \{\int_S g_t d\mu_{a, -}\}_{t \in T} \overset{d}{=} \{\int_S f_t d\mu_{a, -}\}_{t \in T} \), which is equivalent to the fact that \( \{Y_t\}_{t \in T} \) is stationary. The proof of part (ii) is similar and thus omitted. \( \square \)

Observe that not all \( SoS \) processes can be associated to \( \alpha \)-Fréchet processes, since not all \( SoS \) processes have nonnegative spectral representations. For an \( SoS \) process \( \{X_t\}_{t \in T} \) with spectral representation \( \{f_t\}_{t \in T} \) to have an associated \( \alpha \)-Fréchet process, a necessary and sufficient condition is that for all \( t_1, \ldots, t_n \in T, f_{t_1}, \ldots, f_{t_n} \) satisfy the associable condition (3.2). We say such \( SoS \) processes are \( max\)-associable. Now, Proposition 3.2 becomes:

**Theorem 4.2.** Any \( S \alpha S \) process \( \{X_t\}_{t \in T} \) with representation (1.1) is \( max\)-associable, if and only if for all \( t_1, t_2 \in T, \)

\[
f(t_1)s(t_2) \geq 0, \mu \text{-a.e.}
\]  

(4.2)

Indeed, by Theorem 4.2 for any \( max\)-associable spectral representation \( \{f_t\}_{t \in T} \), \( \{f_t\}_{t \in T} \) is also a spectral representation for the same process. Clearly, if the spectral functions are nonnegative, then the \( SoS \) processes are \( max\)-associable. We give two simple examples next.
Example 4.1 (Association of Mixed Fractional Motions). Consider the self-similar $S \alpha S$ processes $\{X_t\}_{t \in \mathbb{R}^+}$ with the following representations
\begin{equation}
\{X_t\}_{t \in \mathbb{R}^+} \overset{d}{=} \left\{ \int_{E} \int_{0}^{\infty} t^{H-1/2} \mathbb{I}_t(x) \right\} M_{\alpha,+}(dx, du)_{t \in \mathbb{R}^+}, \quad H \in (0, \infty),
\end{equation}
(4.3)
where $(E, \mathcal{E}, \nu)$ is a standard Lebesgue space, $M_{\alpha,+}$ is an $S \alpha S$ random measure on $X \times \mathbb{R}^+$ with control measure $m(dx, du) = \nu(dx)du$ and $g \in L^p(E \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}^+)$. Such processes are called mixed fractional motions (see Burnecki et al., 1998). When $\mu \geq 0$ a.e., the process $\{X_t\}_{t \in \mathbb{R}^+}$ is max-associable. The Corollary 4.1 implies the associated $\alpha$-Fréchet process is $H$-self-similar.

Example 4.2 (Association of Chentsov $S \alpha S$ Random Fields). Recall that $\{X_t\}_{t \in \mathbb{R}^n}$ is a Chentsov $S \alpha S$ random field, if
\begin{equation}
\{X_t\}_{t \in \mathbb{R}^n} \equiv \{M_{\alpha,+}(V_t)\}_{t \in \mathbb{R}^n} \overset{d}{=} \left\{ \int_{S} V_t(u)M_{\alpha,+}(du) \right\}_{t \in \mathbb{R}^n}.
\end{equation}
(4.4)
Here, $0 < \alpha < 2$, $(\mathcal{X}, \mu)$ is a measure space and $V_t$, $t \in \mathbb{R}^n$ is a family of measurable sets such that $\mu(V_t) < \infty$ for all $t \in \mathbb{R}^n$ (see Ch. 8 in Samorodnitsky and Taqqu, 1994). Since $V_t(u) \geq 0$, all Chentsov $S \alpha S$ random fields are max-associable.

We conclude this section with some examples of $S \alpha S$ processes that are not max-associate. In particular, recall that the $S \alpha S$ processes with stationary increments (zero at $t = 0$) characterized by dissipative flows shown in Surgailis et al. (1998) have representation
\begin{equation}
\{X_t\}_{t \in \mathbb{R}} \overset{d}{=} \left\{ \int_{E} \int_{\mathbb{R}} (G(x, t + u) - G(x, u))M_{\alpha,+}(dx, du) \right\}_{t \in \mathbb{R}}.
\end{equation}
(4.5)
Here, $(E, \mathcal{E}, \nu)$ is a standard Lebesgue space, $M_{\alpha,+}$, $\alpha \in (0, 2)$, is an $S \alpha S$ random measure with control measure $m(dx, du) = \nu(dx)du$ and $G \in \mathcal{E} \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function such that, for all $t \in \mathbb{R},$
\begin{equation}
G_t(x, u) = G(x, t + u) - G(x, u), \quad x \in \mathcal{E}, u \in \mathbb{R}
\end{equation}
belongs to $L^p(E \times \mathbb{R} \times \mathbb{R})$. The process $\{X_t\}_{t \in \mathbb{R}}$ in (4.4) is called a mixed moving average with stationary increments. The following result provides a partial characterization of the max-associable $S \alpha S$ processes $\{X_t\}_{t \in \mathbb{R}}$, which have the representation (4.4). We shall suppose that $E$ is equipped with a metric $\rho$ and endow $E \times \mathbb{R}$ with the product topology.

Proposition 4.1. Consider on $S \alpha S$ process $\{X_t\}_{t \in \mathbb{R}}$ with representation (4.4). Suppose there exists a closed set $\mathcal{N} \subset E \times \mathbb{R}^n$ such that $m(\mathcal{N}) = 0$ and the function $G$ is continuous at all $(x, u) \in \mathbb{N}^c \equiv E \times \mathbb{R} \setminus \mathcal{N}$, w.r.t. the product topology. Then, $\{X_t\}_{t \in \mathbb{R}}$ is max-associable, if and only if
\begin{equation}
G(x, u) = f(x)1_{A}(u) + c(x), \quad \text{on } \mathbb{N}^c.
\end{equation}
(4.6)
Namely, for all $x \in \mathcal{E}, G(x, u)$ can take at most two values on $\mathbb{N}^c$.

Proof. By Theorem 4.2, $\{X_t\}_{t \in \mathbb{R}}$ is max-associable, if and only if for all $t_1, t_2 \in \mathbb{R},$
\begin{equation}
G_{t_1}(x, u)G_{t_2}(x, u) = (G(x, t_1 + u) - G(x, u))(G(x, t_2 + u) - G(x, u)) \geq 0, \quad \text{m-a.e. } (x, u) \in \mathbb{E} \times \mathbb{R}.
\end{equation}
(4.7)
First, we show the ‘if’ part. Define $\tilde{G}(x, u) := G(x, u)$ (given by (4.5)) on all $-\mathbb{N}^c$ and $\bar{G}(x, u) := f(x)1_{A}(u) + c(x)$ on $\mathbb{N}^c$ (if $A$ and $c(x)$ are not defined, then set $\bar{G}(x, u) = 0$). Set $\tilde{G}(x, u) = \bar{G}(x, u + t) - \bar{G}(x, u)$. Note that $\bar{G}(x, u)$ is another spectral representation of $\{X_t\}_{t \in \mathbb{R}}$ and for all $(x, u)$, $\{1\}_{A}(u + t) - 1\}_{A}(u)\}_{t \in \mathbb{R}}$ can take at most 2 values, one of which is 0. This observation implies (4.6) with $G(x, u)$ replaced by $\bar{G}(x, u)$, whence $\{X_t\}_{t \in \mathbb{R}}$ is max-associable.

Next, we prove the ‘only if’ part. We show that (4.6) is violated, if $G(x, u)$ takes more than 2 different values on $(\mathcal{E} \times \mathbb{R}) \cap \mathbb{N}^c$ for some $x \in \mathcal{E}$. Suppose there exists $x \in \mathcal{E}, u_1, u_2 \in \mathbb{R}$ such that $(x, u_1) \in \mathbb{N}^c$ and $g_{u_1} := G(x, u_1)$ are mutually different, for $i = 1, 2, 3$. Indeed, without loss of generality we may suppose that $g_{u_1} < g_{u_2} < g_{u_3}$. Then, by the continuity of $G$, there exists $\epsilon > 0$ such that $B_{i} := B((x, \epsilon), (u_i, \epsilon)) \cap \mathbb{N}^c \cap \mathcal{E}$ is disjoint with $B((x, \epsilon), (u_i, \epsilon), (u_j, \epsilon))$, $i = 1, 2, 3$ are disjoint sets with $B((x, \epsilon), (u_i, \epsilon)) := \{y \in \mathbb{E} : \rho(x, y) < \epsilon\}$, $\rho$ is the metric on $\mathbb{E}$ and
\begin{equation}
\sup_{B_{i} \cap \mathbb{N}^c} G(x, u) < \inf_{B_{i} \cap \mathbb{N}^c} G(x, u) \leq \sup_{B_{i} \cap \mathbb{N}^c} G(x, u) < \inf_{B_{i} \cap \mathbb{N}^c} G(x, u).
\end{equation}
(4.8)
Put $t_1 = u_1 - u_2$ and $t_2 = u_2 - u_3$. Inequality (4.7) implies that $G_{t_1}(x, u)G_{t_2}(x, u) < 0$ on $B_{2} \cap \mathbb{N}^c$. This, in view of Theorem 4.2, contradicts the max-associability. We have thus shown (4.5). □

We give two classes of $S \alpha S$ processes, which cannot be associated to any $\alpha$-Fréchet processes, according to Proposition 4.1.

Example 4.3 (Non-associability of Linear Fractional Stable Motions). The linear fractional stable motions (see Ch. 7.4 in Samorodnitsky and Taqqu, 1994) have the following spectral representations:
\[ \{X_t\}_{t \in \mathbb{R}} \overset{d}{=} \left\{ \int_{\mathbb{R}} \left[ a \left( (t + u)^{H-1/\alpha} - u^{H-1/\alpha} \right) + b \left( (t + u)^{H-1/\alpha} - u^{H-1/\alpha} \right) \right] M_{\alpha,+}(du) \right\}_{t \in \mathbb{R}}, \]

Here \( H \in (0, 1), \alpha \in (0, 2), H \neq 1/\alpha, a, b \in \mathbb{R} \) and \(|a| + |b| > 0\). By Proposition 4.1, these processes are not max-associable.

**Example 4.4 (Non-associability of Telecom Processes).** The Telecom process offers an extension of fractional Brownian motion consistent with heavy-tailed fluctuations. It is a large scale limit of renewal reward processes and it can be obtained by choosing the distribution of the rewards accordingly (see Levy and Taqqu, 2000 and Pipiras et al., 2004). A Telecom process \( \{X_t\}_{t \in \mathbb{R}} \) has the following representation

\[ \{X_t\}_{t \in \mathbb{R}} \overset{d}{=} \left\{ \int_{\mathbb{R}} \int_{\mathbb{R}} e^{(H-1)/\alpha} \left( F(e^t (t + u)) - F(e^u) \right) M_{\alpha,+}(ds, du) \right\}_{t \in \mathbb{R}}, \]

where \( 1 < \alpha < 2, 1/\alpha < H < 1, F(z) = (z \wedge 0 + 1)_{+}, z \in \mathbb{R} \) and the SoS random measure \( M_{\alpha,+} \) is with control measure \( m_{\alpha}(ds, du) = dsdu \). By Proposition 4.1, the Telecom process is not max-associable.

**Remark 4.2.** It is important that the index \( T \) in Proposition 4.1 is the entire real line \( \mathbb{R} \). Indeed, in both Examples 4.3 and 4.4, when the time index is restricted to the half-line \( \mathbb{R}^{+} \) (or \( \mathbb{R}^{-} \)), the processes \( \{X_t\}_{t \in T} \) satisfy condition (4.2) and are therefore max-associable.

### 5. Association of classifications

In this section, we show how to apply the association technique to relate various classification results for SoS and \( \alpha \)-Fréchet processes. Note that, many classifications of SoS (\( \alpha \)-Fréchet as well) processes are induced by suitable decompositions of the measure space \((S, \mu)\). The following theorem provides an essential tool for translating classification results for SoS to \( \alpha \)-Fréchet processes, and vice versa.

**Theorem 5.1.** Suppose an \( \alpha \)-S process \( \{X_t\}_{t \in \mathbb{R}} \) and an \( \alpha \)-Fréchet process \( \{Y_t\}_{t \in \mathbb{R}} \) are associated by two spectral representations \( \{f_{i_t}^{(0)}\}_{t \in \mathbb{R}} \subset L_{+}^{2}(S, \mu_{i}) \) for \( i = 1, 2 \). That is,

\[ \{X_t\}_{t \in \mathbb{R}} \overset{d}{=} \left\{ \int_{S} f_{i_t}^{(0)} dM_{\alpha,+} \right\}_{t \in \mathbb{R}} \quad \text{and} \quad \{Y_t\}_{t \in \mathbb{R}} \overset{d}{=} \left\{ \int_{S} f_{j_t}^{(0)} dM_{\alpha,\vee} \right\}_{t \in \mathbb{R}}, \quad i = 1, 2. \]

Then, for any measurable subsets \( A_{i} \subset S, i = 1, 2 \), we have

\[ \left\{ \int_{A_{i}} f_{i_t}^{(1)} dM_{\alpha,+} \right\}_{t \in \mathbb{R}} \overset{d}{=} \left\{ \int_{A_{i}} f_{i_t}^{(2)} dM_{\alpha,\vee} \right\}_{t \in \mathbb{R}} \iff \left\{ \int_{A_{i}} f_{j_t}^{(1)} dM_{\alpha,\vee} \right\}_{t \in \mathbb{R}} \overset{d}{=} \left\{ \int_{A_{i}} f_{j_t}^{(2)} dM_{\alpha,+} \right\}_{t \in \mathbb{R}}. \]

The proof follows from Theorem 1.1 by restricting the measures onto the sets \( A_{i} \), \( i = 1, 2 \).

For an \( \alpha \)-S process \( \{X_t\}_{t \in \mathbb{R}} \) with spectral functions \( \{f_{t}\}_{t \in A} \subset L^{2}(S, \mu) \), a decomposition typically takes the form \( \{X_t\}_{t \in \mathbb{R}} \overset{d}{=} \left( \sum_{j=1}^{n} X_t^{(j)} \right)_{t \in \mathbb{R}} \), where \( X_t^{(j)} = \int_{A(t)} f_{t} dM_{\alpha,+} \) for all \( t \in T \) and \( A(t) \), \( 1 \leq j \leq n \) are disjoint subsets of \( S = \bigcup_{j=1}^{n} A(t) \). The components \( \{X_t^{(j)}\}_{t \in \mathbb{R}}, 1 \leq j \leq n \) are independent \( \alpha \)-S processes. When \( \{X_t\}_{t \in \mathbb{R}} \) is max-associable, Theorem 5.1 enables us to define the associated decomposition, for the \( \alpha \)-Fréchet process \( \{Y_t\}_{t \in \mathbb{R}} \) associated with \( \{X_t\}_{t \in \mathbb{R}} \). Namely, we have \( \{Y_t\}_{t \in \mathbb{R}} \overset{d}{=} \left( \bigvee_{j=1}^{n} Y_t^{(j)} \right)_{t \in \mathbb{R}} \), where \( Y_t^{(j)} = \int_{A(t)} f_{t} dM_{\alpha,\vee} \) for all \( t \in T \). Conversely, given a decomposition for \( \alpha \)-Fréchet processes, we can define a corresponding decomposition for the associated \( \alpha \)-S processes.

**Example 5.1 (Conservative–dissipative Decomposition).** In the seminal work, Rosiński (1995) established the conservative–dissipative decomposition for \( \alpha \)-S processes. Namely, for any \( \{X_t\}_{t \in \mathbb{R}} \) with representation (1.1), one has

\[ \{X_t\}_{t \in \mathbb{R}} \overset{d}{=} \left\{ X_t^{C} + X_t^{D} \right\}_{t \in \mathbb{R}}, \]

where \( X_t^{C} = \int_{C} f_{t} dM_{\alpha,+} \) and \( X_t^{D} = \int_{D} f_{t} dM_{\alpha,+} \) for all \( t \in T \), with \( C \) and \( D \) defined by

\[ C := \left\{ s : \int_{T} f_{t}(s)\lambda(dt) = \infty \right\} \quad \text{and} \quad D := S \setminus C. \] (5.1)

When \( \{X_t\}_{t \in \mathbb{R}} \) is stationary, the sets \( C \) and \( D \) correspond to the Hopf decomposition \( S = C \cup D \) of the non-singular flow associated with \( \{X_t\}_{t \in \mathbb{R}} \) (see Rosiński, 1995 for details). Therefore, \( \{X_t^{C}\}_{t \in \mathbb{R}} \) and \( \{X_t^{D}\}_{t \in \mathbb{R}} \) are referred to as the conservative and dissipative components of \( \{X_t\}_{t \in \mathbb{R}} \), respectively. Theorem 5.1 enables us to use (5.1) to establish the parallel decomposition of the associated \( \alpha \)-Fréchet process \( \{Y_t\}_{t \in \mathbb{R}} \). Namely, for the associated \( \{Y_t\}_{t \in \mathbb{R}} \), we have \( \{Y_t\}_{t \in \mathbb{R}} \overset{d}{=} \left( Y_t^{C} \vee Y_t^{D} \right)_{t \in \mathbb{R}} \), where \( Y_t^{C} = \int_{C} |f_{t}| dM_{\alpha,\vee} \) and \( Y_t^{D} = \int_{D} |f_{t}| dM_{\alpha,\vee} \) for all \( t \in T \). This decomposition was established in Wang and Stoev (2009) by using different tools.
Remark 5.1. Similar associations can be established for other decompositions, including positive–null decomposition (see Samorodnitsky, 2005 and Wang and Stoev, 2009), and the decompositions of the above two types for random fields \((T = \mathbb{R}^d\) or \(\mathbb{R}^d\); see Roy and Samorodnitsky, 2008 and Wang et al., 2009). A more specific decomposition for SoS processes with representation (4.4) was developed in Pipiras and Taqqu (2002), and one can obtain the corresponding decomposition for the associated \(\alpha\)-Fréchet process by Theorem 5.1.

6. Discussion

Recently, Kabluchko (2009) introduced a similar notion of association. We became aware of his result toward the end of our work. The two approaches are technically different. Kabluchko’s approach utilizes spectral measures, while ours is based on the structure of max–linear and linear isometries. These two approaches lead to the equivalent notions of association (see Lemma 2 in Kabluchko, 2009). For example, our Corollary 4.1 can also be obtained following his approach. On the other hand, our approach leads to a more direct proof of the following, which is Lemma 3 in Kabluchko (2009).

Lemma 6.1. Let \(\{X_t\}_{t \in T}\) be an S a S process and \(\{Y_t\}_{t \in T}\) be an \(\alpha\)-Fréchet process. Suppose \(\{X_t\}_{t \in T}\) and \(\{Y_t\}_{t \in T}\) are associated by \(\{f_t\}_{t \in T} \subset L^2(S, \mu)\). Then for any \(t_1, t_2, \ldots, t_n \in T\), as \(n \to \infty\), \(X_{t_n}\) converges in probability to \(X_t\), if and only if \(Y_{t_n}\) converges in probability to \(Y_t\).

Proof. By Proposition 3.5.1 in Samorodnitsky and Taqqu (1994), \(X_{t_n} \overset{p}{\to} X_t\) as \(n \to \infty\), and if only if \(\|f_{t_n} - f_t\|_{L^2(S, \mu)} \to 0\) as \(n \to \infty\). This is equivalent to, by Theorem 2.1 and Lemma 2.3 in Stoev and Taqqu (2006), \(Y_{t_n} \overset{p}{\to} Y_t\) as \(n \to \infty\). \(\square\)

Kabluchko (2009) also proved (Theorem 9 therein) that an \(\alpha\)-Fréchet process is mixing (ergodic resp.) if and only if the associated SoS process is mixing (ergodic resp.). The proofs of these results, however, are not simple consequences of the notion of association. By association one can easily obtain new classes of \(\alpha\)-Fréchet processes, while the probabilistic properties of the new processes (e.g. the associated \(\alpha\)-Fréchet processes in examples in Section 4), however, do not automatically follow ‘by association’ and are yet to be investigated.

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Appendix. Proofs of auxiliary results

We first need the following lemma.

Lemma A.1. If \(F \subset L^2_+ (S, \mu)\), then

(i) \(\rho(F) = \rho(\text{span}_+ (F)) = \rho(\text{span}_- (F))\), and

(ii) for any \(f^{(1)} \in \text{span}_+ (F)\) and \(f^{(2)} \in \text{span}_- (F)\), \(f^{(1)}/f^{(2)} \in \rho(F)\).

Proof. (i) First, for any \(f_i, g_i \in F\), \(a_i \geq 0, b_i \geq 0, i \in \mathbb{N}\), we have

\[
\bigg\{ \frac{\sum_{i \in \mathbb{N}} a_i f_i}{\sum_{j \in \mathbb{N}} b_j g_j} \leq x \bigg\} = \bigcap_{i \in \mathbb{N}} \bigg\{ \frac{a_i f_i}{b_i g_i} \leq x \bigg\} = \bigcap_{i \in \mathbb{N}} \bigcup_{j \in \mathbb{N}} \bigg\{ \frac{a_i f_i}{b_j g_j} < x + \frac{1}{k} \bigg\}
\]

hence \(\rho(\text{span}_+ (F)) \subset \rho(\text{span}_- (F))\).

To show \(\rho(\text{span}_+ (F)) \subset \rho(\text{span}_- (F))\), we shall first prove that \(\rho(\text{span}_+ (F)) \subset \rho(\text{span}_- (F))\), where span_+ (F) involves only finite positive-linear combinations. For all \(f_1, f_2, g_1 \in F, a_1, b_1, b_2 \geq 0, \) we have

\[
\frac{a_1 f_1 + a_2 f_2}{b_1 g_1} \leq x \implies \bigg\{ \frac{a_1 f_1}{b_1 g_1} \leq q \bigg\} \cap \bigg\{ \frac{a_2 f_2}{b_1 g_1} \leq x - q \bigg\}
\]

This shows that \((a_1 f_1 + a_2 f_2)/b_1 g_1\) is \(\rho(\text{span}_- (F))\) measurable. By using the fact that \(F\) contains only nonnegative functions and since \(\frac{b_1 g_1}{a_1 f_1 + a_2 f_2} \leq x \implies \frac{a_1 f_1 + a_2 f_2}{b_1 g_1} \geq \frac{1}{x}\), for \(x > 0\), we similarly obtain that \((a_1 f_1 + a_2 f_2)/(b_1 g_1 + b_2 g_2)\) is \(\rho(\text{span}_+ (F))\) measurable. Similarly arguments can be used to show that \((\sum_{i=1}^n a_i f_i)/(\sum_{i=1}^n b_i g_i)\) is \(\rho(\text{span}_- (F))\) measurable for all \(a_i, b_i \geq 0, f_i, g_i \in F, 1 \leq i \leq n\).

We have thus shown that \(\rho(\text{span}_+ (F)) \subset \rho(\text{span}_- (F))\). If now \(f, g \in \text{span}_+ (F)\), then there exist two sequences \(f_n, g_n \in \text{span}_+ (F)\), such that \(f_n \to f\) and \(g_n \to g\) a.e. Thus, \(h_n := f_n/g_n \to h := f/g\) as \(n \to \infty\), a.e. Since \(h_n\) are \(\rho(\text{span}_+ (F))\) measurable for all \(n \in \mathbb{N}\), so is \(h\). Hence \(\rho(\text{span}_+ (F)) = \rho(\text{span}_- (F)) \subset \rho(\text{span}_- (F))\).
(ii) By the previous argument, it is enough to focus on finite linear and max-linear combinations. Suppose \( f^{(1)} = \sum_{i=1}^{n} a_{fi} \) and \( f^{(2)} = \bigvee_{j=1}^{p} b_{gj} \) for some \( f_i, g_j \in F \), \( a_i, b_j \geq 0 \). Then, for all \( x > 0 \),
\[
\left\{ \frac{n}{n} \sum_{i=1}^{n} a_{fi} \right\} \leq \left\{ \left( \bigvee_{j=1}^{p} b_{gj} \right) \right\} = x \subseteq \rho(F).
\]
It follows that \( f^{(1)}/f^{(2)} \in \rho(F) \).

**Proof of Proposition 3.1.** First we show \( \mathcal{R}_{e,+}(\mathcal{F}_-) \supseteq \mathcal{R}_{e,+}(\mathcal{F}_+) \), where \( \mathcal{F}_- \) and \( \mathcal{F}_+ \) are defined in (3.1). By (2.2), it suffices to show that, for any \( r_2 \in \rho(\mathcal{F}_-) \), \( f^{(2)} \in \mathcal{F}_+ \), there exist \( r_1 \in \rho(\mathcal{F}_-) \) and \( f^{(1)} \in \mathcal{F}_- \) such that
\[
r_1 f^{(1)} = r_2 f^{(2)}.
\]
To obtain (A.1), we need the concept of full support. We say a function \( g \) has full support in \( F \) (an arbitrary collection of functions defined on \( (S, \mu) \)), if \( g \in F \) and for all \( f \in F \), \( \mu(\operatorname{supp}(g) \setminus \operatorname{supp}(f)) = 0 \). Let \( \operatorname{supp}(f) := \{ s \in S : f(s) \neq 0 \} \).

By Lemma 3.2 in Wang and Stoev (2009), there exists function \( f^{(1)} \in \mathcal{F}_- \), which has full support in \( \mathcal{F}_+ \). One can show that this function has also full support in \( \mathcal{F}_+ \). Indeed, let \( g \in \mathcal{F}_+ \) be arbitrary. Then, there exist \( g_n = \sum_{i=1}^{n} a_n g_{n,i} \), \( a_n \geq 0 \) and \( g_{n,i} \in F \subset \mathcal{F}_- \) such that \( g_n \to g \) as \( n \to \infty \). Note that \( \mu(\operatorname{supp}(g_{n,i}) \setminus \operatorname{supp}(f)) = 0 \) for all \( n \). Thus, for all \( \epsilon > 0 \), we have
\[
\mu(\{ g_n - g > \epsilon \}) \geq \mu(\{ |g_n| > \epsilon \} \setminus \operatorname{supp}(f)).
\]
Since \( \mu(\{|g_n - g| > \epsilon \}) \rightarrow 0 \) as \( n \to \infty \), it follows that \( \mu(\{|g_n| > \epsilon \} \setminus \operatorname{supp}(f)) = 0 \) for all \( \epsilon > 0 \). Let \( g \in \mathcal{F}_- \) be arbitrary.

Now, set \( r_1 := r_2 f^{(1)}/f^{(2)} \). We have (A.1). (Note that \( f^{(2)} = 0 \), \( \mu \)-a.e. on \( S \).) By setting \( 0/0 = 0 \), \( f^{(2)}/f^{(1)} \) is well-defined. Lemma A.1 (iii) implies that \( f^{(2)}/f^{(1)} \in \rho(F) \), whence \( r_1 \in \rho(F) = \rho(\mathcal{F}_+) \). We have thus shown \( \mathcal{R}_{e,+}(\mathcal{F}_-) \supseteq \mathcal{R}_{e,+}(\mathcal{F}_-) \). In a similar way one can show \( \mathcal{R}_{e,+}(\mathcal{F}_+) \subset \mathcal{R}_{e,+}(\mathcal{F}_+) \). □

**Proof of Proposition 3.2.** First, suppose (3.2) does not hold but (1.5) holds. Then, without loss of generality, we can assume that there exists \( S_1^{(i)} \subset S_1 \) such that \( f^{(1)}(s) > 0 \), \( f^{(2)}(s) < 0 \) for all \( s \in S_1^{(i)} \). It follows from (1.5) that there exists a linear isometry \( U \) such that, by Theorem 2.1, \( U f^{(1)} = f^{(1)} \). This contradicts the fact that \( f^{(2)}/f^{(1)} \) is both nonnegative on \( S_2 \).

On the other hand, suppose (3.2) is true. Define \( U f^{(1)} := |f^{(1)}| \). It follows from (3.2) that \( U \) can be extended to a positive-linear isometry from \( L^\infty_+(S_1, \mu_1) \) to \( L^\infty_+(S_2, \mu_2) \), which implies (1.5). □

References


