1. Suppose that traffic on a road follows a Poisson process with rate $\lambda$ cars per minute. A chicken needs a gap of length at least $c$ minutes in the traffic to cross the road. To compute the time the chicken will have to wait to cross the road, let $t_1, t_2, t_3, \ldots$ be the interarrival times for the cars and let $J = \min\{j : t_j > c\}$. If $T_n = t_1 + \cdots + t_n$, then the chicken will start to cross the road at time $T_{J-1}$ and complete his journey at time $T_{J-1} + c$.

(a) [4 points]. Suppose $T$ is exponentially distributed with rate $\lambda$. Find $\mathbb{E}[T \mid T < c]$.

Hint: Using the identity $\mathbb{E}[T] = \mathbb{P}(T < c) \mathbb{E}[T \mid T < c] + \mathbb{P}(T > c) \mathbb{E}[T \mid T > c]$ leads to a nice solution, though you can also solve the problem by direct calculation.

Solution:

$$\mathbb{E}(T \mid T < c) = \frac{\mathbb{E}(T \mid T < c)}{\mathbb{P}(T < c)} = \frac{\int_0^c t \lambda e^{-\lambda t} dt}{1 - e^{-c}} = \frac{-c e^{-\lambda c} + \frac{1}{\lambda} (1 - e^{-\lambda c})}{1 - e^{-c}}.$$ 

(b) [6 points] Use part (a) to show $\mathbb{E}(T_{J-1} + c) = (e^{\lambda c} - 1)/\lambda$. If you have not solved (a), you may leave your answer in terms of $\mathbb{E}[T \mid T < c]$.

Solution: Note that

$$T_{J-1} + c = c \mathbb{I}_{(t_1 > c)} + (t_1 + c + \sum_{k=2}^{J-1} t_k) \mathbb{I}_{(t_1 \leq c)}.$$ 

Taking expectation on both sides above, we have

$$\mathbb{E}(T_{J-1} + c) = c \mathbb{P}(t_1 > c) + \mathbb{P}(t_1 \leq c) \mathbb{E}(t_1 \mid t_1 \leq c) + \mathbb{P}(t_1 \leq c) \mathbb{E} \left( \sum_{k=2}^{J-1} t_k + c \mid t_1 \leq c \right). \quad (1)$$

Note that in above, by result in (a), the second term on the r.h.s. of (1) equals $-c e^{-\lambda c} + \frac{1}{\lambda} (1 - e^{-\lambda c})$, and the third term equals $\mathbb{P}(t_1 \leq c) \mathbb{E}(T_{J-1} + c)$ by memoryless property. Hence, it follows that

$$\mathbb{E}(T_{J-1} + c) = \frac{c e^{-\lambda c} + (-c e^{-\lambda c} + \frac{1}{\lambda} (1 - e^{-\lambda c}))}{\mathbb{P}(t_1 > c)}$$

$$= \frac{\frac{1}{\lambda} (1 - e^{-\lambda c})}{e^{-\lambda c}} = \frac{1}{\lambda} \left( e^{\lambda c} - 1 \right).$$

2. We investigate a martingale solution to the same situation problem from Question 1. As before, traffic on a road follows a Poisson process with rate $\lambda$ cars per minute. A chicken needs a gap of length at least $c$ minutes in the traffic to cross the road. $t_1, t_2, t_3, \ldots$ are the interarrival times for the cars and $J = \min\{j : t_j > c\}$. If $T_n = t_1 + \cdots + t_n$, then the chicken will start to cross the road at time $T_{J-1}$ and complete his journey at time $T_{J-1} + c$. Note that $T_n - (n/\lambda)$ is a martingale.

(a) [3 points] Argue that $J$ is a stopping time for $t_1, t_2, \ldots$, and explain why $J - 1$ is not a stopping time.
Solution: Let \( X_n = T_n - \frac{n}{\lambda} \). \( J \) is a stopping time because

\[
\{ J = n \} = \{ T_1 < c, T_2 - T_1 < c, \ldots, T_{n-1} - T_{n-2} < c, T_n - T_{n-1} > c \}
\]

\[
= \{ X_1 + \frac{1}{\lambda} < c, X_2 - X_1 + \frac{1}{\lambda} < c, \ldots, X_{n-1} - X_{n-2} + \frac{1}{\lambda} < c X_n - X_{n-1} > c \},
\]

which is determined by the value of \( X_1, \ldots, X_n \). Similarly, \( \{ J - 1 = n \} = \{ J = n + 1 \} \) is determined by the value of \( X_1, \ldots, X_{n+1} \). Thus, \( J \) is a stopping time, but \( J - 1 \) is not.

(b) [7 points] Use a martingale argument to show that \( \mathbb{E}(T_{J-1} + c) = (e^{\lambda c} - 1)/\lambda \).

Solution: First, we have

\[
\mathbb{E}(T_{J-1} + c) = \mathbb{E}(T_J - t_J + c) = \mathbb{E}(T_J - \frac{J}{\lambda} + \frac{J}{\lambda} - t_J + c) = \mathbb{E}(T_J - \frac{J}{\lambda}) + \frac{\mathbb{E}J}{\lambda} - \mathbb{E}(t_J) + c.
\]

We calculate the three expectations in the r.h.s. above respectively. It is easy to see that \( J \) has geometric distribution with parameter \( p = P(t_1 > c) = e^{-\lambda c} \). Then \( \mathbb{E}J = e^{\lambda} \). Also,

\[
\mathbb{E}(t_J) = \mathbb{E}(t_1|t_1 > c) = c + \mathbb{E}(t_1) = c + \frac{1}{\lambda}.
\]

To calculate \( \mathbb{E}(T_J - \frac{J}{\lambda}) \), since we have shown that \( X_n = T_n - \frac{n}{\lambda} \) is a martingale and \( J \) is a stopping time, by the fact that \( \mathbb{E}J < \infty \) and that

\[
\mathbb{E}(|X_{n+1} - X_n| |X_1, \ldots, X_n) = \mathbb{E}\left(|t_1 - \frac{1}{\lambda}| |X_1, \ldots, X_n\right) \leq \mathbb{E}(t_1) + \frac{1}{\lambda} < \infty,
\]

we apply the martingale stopping theorem to obtain

\[
\mathbb{E}(T_J - \frac{J}{\lambda}) = \mathbb{E}(T_1 - \frac{1}{\lambda}) = 0.
\]

Plugging in, we have

\[
\mathbb{E}(T_{J-1} + c) = 0 + \frac{e^{\lambda c}}{\lambda} - c - \frac{1}{\lambda} + c = \frac{e^{\lambda c} - 1}{\lambda}.
\]

3. We study a queue with impatient customers. Customers arrive at a single server as a Poisson process with rate \( \lambda \) and require an exponential amount of service with rate \( \mu \). Customers waiting in line are impatient and if they are not in service they will leave at rate \( \delta \) independent of their position in the queue. Show that for any \( \delta > 0 \) the system has a stationary distribution, and find an expression for this distribution.

Solution: Let \( X(t) \) denote the process. It has states \( \{0, 1, 2, \ldots\} \). We can write the transition matrix as follows.

\[
\begin{align*}
q_{i+1} & = \lambda \\
q_{i-1} & = \mu + (i - 1)\delta, \quad i \geq 1 \\
qu_i & = 0, \quad \text{otherwise}
\end{align*}
\]

It is clear that we have a death and birth process. Then, the stationary distribution \( P_i \) satisfies:

\[
\sum_{i=0}^{\infty} P_i = 1 \quad \text{and} \quad P_i q_{i+1} = P_{i+1} q_{i+1, i}.
\]

(2)
By solving (2), we have

\[ P_i = \frac{\lambda^i}{\prod_{j=0}^{i-1} (\mu + j\delta)} P_0, \quad P_0 = \left(1 + \sum_{i=1}^{\infty} \frac{\lambda^i}{\prod_{j=0}^{i-1} (\mu + j\delta)}\right)^{-1}. \]

To see the existence of stationary distribution, one can check that \( P_0 > 0 \). Indeed, since \( \lambda > 0 \), there exists \( J \in \mathbb{N} \) such that \( \mu + J\delta > \lambda \). Then, it follows that

\[ \sum_{i=1}^{\infty} \frac{\lambda^i}{\prod_{j=0}^{i-1} (\mu + j\delta)} < \sum_{i=1}^{J-1} \frac{\lambda^i}{\prod_{j=0}^{i-1} (\mu + j\delta)} + \sum_{i=J}^{\infty} \left(\frac{\lambda}{\mu + J\delta}\right)^i < \infty. \]

4. A cocaine dealer is standing on a street corner. Customers arrive at times of a Poisson process with rate \( \lambda \). The customer and the dealer then disappear from the street for an amount of time with distribution \( G \) while the transaction is completed. Customers that arrive during this time go away never to return.

(a) [5 points] At what rate does the dealer make sales? Explain your reasoning.

**Solution:** We can define a renewal process, each cycle ending when a transaction is finished. Let \( X_i \) denote the length of cycle \( i \). Let \( T \) be a random variable with distribution \( G \). Then,

\[ \mathbb{E}(X_i) = \frac{1}{\lambda} + \mathbb{E}(T). \]

The long-run rate of transactions is then \( \frac{1}{\mathbb{E}(X_1)} \).

(b) [5 points] What fraction of customers are lost? Explain your reasoning.

**Solution:** The fraction of customers that are lost is the stationary distribution of state 1, i.e., \( \frac{\mathbb{E}(T)}{\frac{1}{\lambda} + \mathbb{E}(T)} \).

5. Let \{\( Z(t), 0 \leq t \leq 1 \}\} be a Brownian bridge, i.e., a Gaussian diffusion with \( \mathbb{E}[Z(t)] = 0 \) and \( \text{Cov}(Z(s), Z(t)) = s \wedge t - st. \) Define \( X(t) = (1 + t)Z(t/(1 + t)) \). Show that \( \{X(t), t \geq 0\} \) is a standard Brownian motion.

**Solution:** Since \( Z(t) \) is a Brownian bridge, it is a Gaussian diffusion. Then \( X(t) \) is a Gaussian diffusion. Now we prove

\[ \mathbb{E}(X(t)) = 0 \quad \text{and} \quad \text{Cov}(X(s), X(t)) = s \wedge t - st. \]

First,

\[ \mathbb{E}(X(t)) = (1 + t)\mathbb{E}\left(Z\left(\frac{t}{1+t}\right)\right) = (1 + t)\mathbb{E}\left(B\left(\frac{t}{1+t}\right) - \frac{t}{1+t}B(1)\right) \]

\[ = (1 + t)\left(0 - \frac{t}{1+t}\right) = 0. \]

Next, by the fact that \( Z(t) \) is a Brownian bridge,

\[ \text{Cov}(X(s), X(t)) = \text{Cov}\left((1 + s)Z\left(\frac{s}{1+s}\right), (1 + t)Z\left(\frac{t}{1+t}\right)\right) \]

\[ = (1 + s)(1 + t)\frac{s \wedge t - st}{1 + s(1 + t)} \]

\[ = s(1 + t) \wedge t(1 + s) - st = s \wedge t - st. \]

We have thus proved that \( X(t) \) is a standard Brownian motion.