7. Brownian Motion & Diffusion Processes

- A continuous time stochastic process with (almost surely) continuous sample paths which has the Markov property is called a **diffusion**.
- "almost surely" means "with probability 1", and we usually assume all sample paths are continuous.
- The simplest and most fundamental diffusion process is Brownian motion $\{B(t)\}$, which is sometimes called the Wiener process $\{W(t)\}$.

Definition 1. $\{B(t)\}$ is **Brownian motion** if it is a diffusion process satisfying

(i) $B(0) = 0$,

(ii) $\mathbb{E}\big[B(t)\big] = 0$ and $\mathrm{Var}\big[B(t)\big] = \sigma^2 t$,

(iii) $\{B(t)\}$ has stationary, independent increments.

Applications of Diffusion Processes

- Many physical processes are continuous in space and time. If all important variables are included in the state of the system, then the future evolution of the system should depend on the current state, i.e., the system is Markovian.
- Any system with these properties is a diffusion, by definition!

  Examples: Molecular motion, stock market fluctuations, communications systems, neurophysiological processes.
- Discrete processes may be well approximated by diffusions, in a limit as the discretization becomes fine.

  Examples: population growth models, disease models, queuing models for large systems.

Brownian Motion as a limit of random walks

- Einstein (1905) showed how the motion of pollen particles in water (Brown, 1827) could be explained by a random walk due to random bombardment of the pollen by water molecules.
- Set $X_1, X_2, \ldots$ iid with $\mathbb{E}[X_1] = 0$, $\text{Var}(X_1) = \sigma^2$. Define $B^{(n)}(t)$ when t a multiple of $1/n$ by

$$B^{(n)}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{nt} X_i,$$

using linear interpolation to define $B^{(n)}(t)$ between these times. Then,

- $B^{(n)}(0) = 0$.
- $\mathbb{E}[B^{(n)}(t)] = 0$ and $\lim_{n\to\infty} \text{Var}[B^{(n)}(t)] = \sigma^2 t$.
- $\{B^{(n)}(t)\}$ has stationary, independent increments at the discrete times, $t = k/n$ (and hence, also, a discrete-time Markov property).
- $\{B^{(n)}(t)\}$ has continuous sample paths.
- One might expect Brownian motion in the limit $n \to \infty$. This was proved by Wiener ($\approx$ 1915).

- Now apply the central limit theorem to $B^{(n)}(t) - B^{(n)}(s)$:

Definition 2. $\{B(t)\}$ is Brownian motion if

(i) $B(0) = 0$,

(ii) $\{B(t)\}$ has stationary, independent increments,

(iii) $B(t) \sim N\big[0, \sigma^2 t\big]$.

- Note: independent increments imply the Markov property.
- Note: it can be shown that Definition 2 implies almost surely continuous sample paths.
- Note: continuous sample paths lead to Gaussian increments much as counting processes lead to Poisson increments.

Hitting times and maximum of $\{B(t)\}$

- $\{B(t)\}$ is symmetric, i.e., $\{-B(t)\}$ is also Brownian motion and $\mathbb{P}[B(t) > 0] = 1/2$. This has useful consequences, for example:

Proposition. $\mathbb{P}\big[\max_{0\leq s\leq t} B(s) \geq a\big] = 2\mathbb{P}\big[B(t) \geq a\big]$

Proof

- Symmetry is also useful to study $Z(t) = |B(t)|$. $\{Z(t)\}$ is called **reflected Brownian motion**. Why?

- Find $\mathbb{P}\big[Z(t) \leq y\big]$ and hence $\mathbb{E}[Z(t)]$, $\mathrm{Var}[Z(t)]$.

The joint distribution of Brownian motion and its maximum

- Let $M(t) = \sup_{0 \le s \le t} B(s)$. Show that, for $y \ge x$,

$$\mathbb{P}\big[M(t) > y, B(t) < x\big] = \int_{2y-x}^{\infty} \frac{1}{\sqrt{2\pi t}} \exp\left\{\frac{-u^2}{2t}\right\} du.$$

Gaussian Processes and Gaussian Diffusions

- $Y = (Y_1, \ldots, Y_n)$ is **multivariate normal** if $Y = \mu + AZ$ where μ is a vector in $\mathbb{R}^n$, A is an $n \times m$ matrix and $Z = (Z_1, \ldots, Z_m)$ is a random vector of iid standard normal variables. We write $Y \sim N\big[\mu, AA^T\big]$ where $\mu = \mathbb{E}[Y]$ and $AA^T = \mathrm{Var}(Y) = \mathbb{E}\Big[\big(Y - \mathbb{E}[Y]\big)\big(Y - \mathbb{E}[Y]\big)^T\Big]$.

- $\{X(t), t \geq 0\}$ is a **Gaussian process** if $(X(t_1), X(t_2), \ldots, X(t_n))$ is multivariate Normal for all $t_1, \ldots, t_n$.

- Since the multivariate normal distribution is specified by its mean and covariance matrix, a Gaussian process is specified by its **mean function** $\mu(t) = \mathbb{E}[X(t)]$ and **covariance function** $\gamma(s,t) = \mathrm{Cov}(X(s), X(t))$

- Gaussian processes are not usually diffusions. Why?

Brownian motion as a Gaussian process:

$\{B(t)\}$ is a Gaussian process, from Definition 2.

Clearly, $\mu(t) = 0$. Find $\gamma(s, t)$.

Conditioning a diffusion on its future value

- Let $\{X(t)\}$ be a diffusion. Let $\{Z(t), 0 \leq t \leq 1\}$ correspond to $X(t)$ conditioned on $X(1) = \alpha$. Show that $\{Z(t)\}$ is a diffusion.

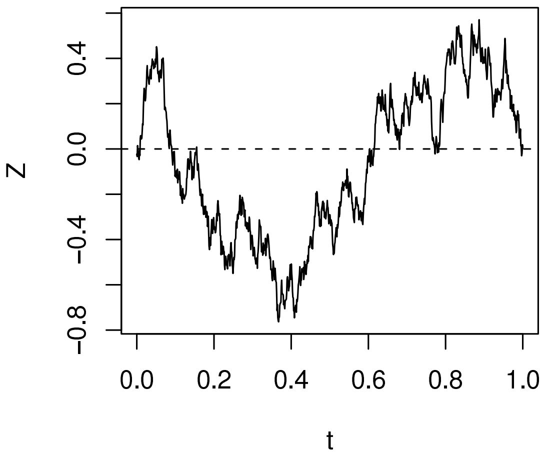
- Note that a **homogeneous diffusion**, where the transition probabilities do not depend on time, becomes **inhomogeneous** once conditioned on a future value.

Conditioning a Gaussian process on its future value

- Let $\{X(t)\}$ be a Gaussian process. Let $\{Z(t), 0 \leq t \leq 1\}$ correspond to $\{X(t)\}$ conditioned on $X(1) = \alpha$. Show that $\{Z(t)\}$ is a Gaussian process.

The Brownian bridge

Let $\{Z(t), 0 \leq t \leq 1\}$ have the distribution of $\{B(t)\}$ conditioned on $B(1) = 0$, where $\{B(t)\}$ is standard Brownian motion. Then $\{Z(t)\}$ is a **Brownian bridge**.



- $\{Z(t)\}$ is a Gaussian diffusion process. Find its mean and covariance functions.

Solution

Solution continued

Another construction of the Brownian bridge

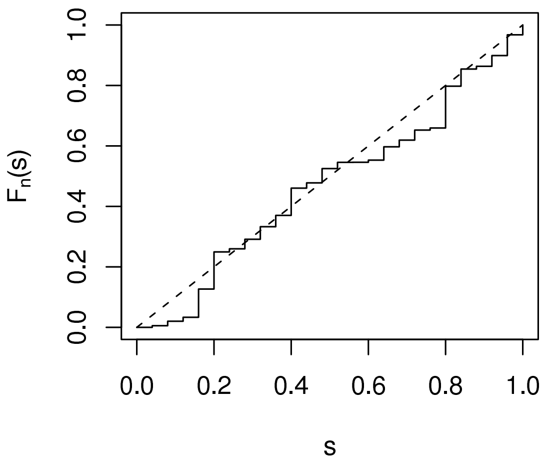
For $\{B(t)\}$ standard Brownian motion, set $Z(t) = B(t) - tB(1)$. Show that $\{Z(t)\}$ is a Brownian bridge.

Solution

An application of the Brownian bridge

- Suppose we suspect that $X_1, \ldots, X_n \sim \text{iid}\, G$, then $U_1, \ldots, U_n \sim \text{iid}\, U[0,1]$ for $U_i = G(X_i)$. Define $F_n(s) = \frac{1}{n}\sum_{i=1}^{n} I_{\{U_i \le s\}}$, called the **empirical distribution function**.

- If it is true that $X_1, \ldots, X_n \sim \text{iid}\, G$, then we expect $F_n(s) \approx s$.



- $F_n(s) - s$ is like a random walk tied at $s = 0$ and $s = 1$, so it may not be surprising that (suitably rescaled) the limit is a Brownian motion tied at $s = 0$ and $s = 1$, i.e. a Brownian bridge.

- It can be shown that $\sqrt{n}\big(F_n(s) - s\big)$ converges to a Brownian bridge. This gives the asymptotic distribution for the Kolmogorov-Smirnov statistic $\sqrt{n}\sup_s |F_n(s) - s|$ to test whether $X_1, \ldots, X_n \sim \text{iid}\, G$.

R code for a Brownian bridge

```
np=1000
t=seq(from=0,to=1,length=np)
X=cumsum(rnorm(np,mean=0,sd=sqrt(1/np)))
Z=X-t*X[np]
plot(t,Z,type="l")
abline(h=0,lty="dashed")
```

R code for the empirical cdf

```
np=25
t=seq(np)/np
U=runif(np)
F=sort(U)
plot(x=c(0,rep(t,each=2),1,1),
     y=c(0,0,rep(F,each=2),1),
     xlab='s',
     ylab=expression(paste(F[n],"(s)")),
     ty="l")
lines(c(0,1),c(0,1),lty="dashed")
```

Transition density of Brownian motion

- The conditional distribution function $F(y,t\,|\,x,s) = \mathbb{P}\big[B(t) \le y \,|\, B(s){=}x\big]$ has corresponding density (for $\sigma^2 = 1$) of $f(y,t\,|\,x,s) = \frac{1}{\sqrt{2\pi(t-s)}} \exp\left\{-\frac{(y-x)^2}{2(t-s)}\right\}$.

- The **forward equation** is $\boxed{\frac{\partial f}{\partial t} = \frac{1}{2}\frac{\partial^2 f}{\partial y^2}}$

  Check this.

- The forward equation is the heat equation in physics, with $f(y,t)$ giving the temperature at location y along a uniform metal bar at time t. The initial conditions for this solution correspond to a pulse of heat injected at x at time $s < t$.

- The **backward equation** for standard Brownian motion is $\boxed{\frac{\partial f}{\partial s} = -\frac{1}{2}\frac{\partial^2 f}{\partial x^2}}$
- We can derive similar equations for general diffusion $\{X(t)\}$. It turns out that the forward and backward equations only depend on the **infinitesimal mean** or **drift**

$$\mu(x,t) = \lim_{\delta t \to 0} \frac{\mathbb{E}\big[X(t+\delta t) - X(t) \,|\, X(t){=}x\big]}{\delta t}$$

and the **infinitesimal variance**

$$\sigma^2(x,t) = \lim_{\delta t \to 0} \frac{\mathbb{E}\big[\big(X(t+\delta t) - X(t)\big)^2 \,|\, X(t){=}x\big]}{\delta t}.$$

- Assuming sufficient regularity for the above limits to exist and for the forward and backward equations to have a unique solution (subject to appropriate boundary conditions), **a diffusion is fully specified by its infinitesimal mean and variance**

Transition densities for general diffusions

- Let $\{X(t)\}$ be a diffusion with infinitesimal mean $\mu(x,t)$ and infinitesimal variance $\sigma^2(x,t)$. Assuming sufficient regularity, the transition density $f(y,t\,|\,x,s)$ of $X(t)$ given $X(s)=x$ satisfies the **backward equation**,

  $\frac{\partial f}{\partial s} = -\mu(x,s)\frac{\partial f}{\partial x} - \frac{1}{2}\sigma^2(x,s)\frac{\partial^2 f}{\partial x^2}$,

  and the **forward equation**,

  $\frac{\partial f}{\partial t} = -\frac{\partial}{\partial y}[\mu(y,t)f] + \frac{1}{2}\frac{\partial^2}{\partial y^2}[\sigma^2(y,t)f]$.

- These equations may have explicit solutions in simple cases (e.g., Gaussian diffusions) and can be solved numerically in other cases.

- The method of constructing a differential equation (often by conditioning and taking a limit) is also useful in other situations, e.g., for hitting probabilities and expected hitting times.

The backward equation can be derived based on the identity

$$f(y,t\,|\,x,s) = \mathbb{E}\big[f\big(y,t\,|\,X(s+h),s+h\big)\,|\,X(s)=x\big]$$

followed by a Taylor series expansion about $h=0$.

Solution

Example. Calculate the infinitesimal mean and variance for a Brownian bridge, i.e., a Gaussian diffusion $\{Z(t), 0 \leq t \leq 1\}$ with $\mathbb{E}[Z(t)] = 0$ and $\mathrm{Cov}(Z(s), Z(t)) = s \wedge t - st$.

- Since a diffusion is specified by its infinitesimal parameters (plus boundary conditions), these give a useful description of the process.

	$\mu(x,t)$	$\sigma^2(x,t)$
Brownian Motion	0	σ^2
Standard Brownian Motion	0	1
Brownian Motion with Drift	μ	σ^2
Brownian Bridge	$-\dfrac{x}{1-t}$	1
Ornstein-Uhlenbeck Process	$-\alpha\, x$	σ^2
Branching Process	$\alpha\, x$	$\beta\, x$
Reflected Brownian Motion	0	σ^2

- Here, $\alpha > 0$ and $\beta > 0$. The branching process is a diffusion approximation based on matching moments to the Galton-Watson process.

- Locally in space and time, the infinitesimal mean & variance are approximately constant, so all diffusions look locally like Brownian motion with drift (except for boundary behavior).

Martingales to Study Brownian Motion

- Since martingale arguments are useful for random walks, we expect them to help for the continuous time analog, i.e., Brownian motion.
- We assume that the stopping theorem also applies in continuous time.

(a) Note that $B(t)$ is itself a martingale.

(b) $[B(t)]^2 - t$ is a martingale. Why?

(c) $\exp\{cB(t) - c^2t/2\}$ is a martingale for any c. Why?

Hitting Times for Brownian Motion with Drift

- $X(t) = B(t) + \mu t$ is called **Brownian motion with drift**. Here, we take $\{B(t)\}$ to be standard Brownian motion, $\sigma^2 = 1$.

- Let $T = \min\{t : X(t) = A \text{ or } X(t) = -B\}$. The random walk analog of T was important for queuing and insurance ruin problems, so T is important if such processes are modeled as diffusions.

- (i) $X(t) - \mu t$ is a martingale.
  Also, $\exp\left\{c(X(t) - \mu t) - c^2 t/2\right\}$ is a martingale.
  Taking $c = -2\mu$ gives
  (ii) $\exp\{-2\mu X(t)\}$ is a martingale.

- We can use (i) and (ii) to find $\mathbb{E}\big[T\big]$ and $\mathbb{P}\big[X(T)=A\big]$ via the martingale stopping theorem.

- Because of continuous sample paths, there is no issue of overshoot or undershoot. Unlike the random walk case, the Martingale solution is exact.

- $E(T)$ and $\mathbb{P}[X(T)\!=A]$ can also be calculated by setting up & solving appropriate differential equations (see Ross, Section 8.4).

- Also, these quantities can be found by taking a limit of random walks. Let $Y_1, Y_2, \ldots$ be iid with $\mathbb{E}[Y_i] = 0$, $\mathrm{Var}(Y_i) = 1$ and let $X^{[n]}(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{nt} (Y_i + \frac{\mu}{\sqrt{n}})$ for $t = k/n$, using linear interpolation for $t \neq k/n$.

- The random walk converges to a diffusion process, and we can check the infinitesimal mean and variance of the limit to show that $X^{[n]}(t)$ converges in distribution to Brownian motion with drift, $X(t)$.

- Let $X_k^{[n]} = \frac{1}{\sqrt{n}} \sum_{i=1}^{k} \left(Y_i + \frac{\mu}{\sqrt{n}}\right)$ and let $T^{[n]} = \min\left\{k : X_k^{[n]} \geq A \text{ or } X_k^{[n]} \leq -B\right\}$.
Then,
$\mathbb{E}\left[T\right] = \lim_{n\to\infty} \mathbb{E}\left[T^{[n]}\right]/n$
and
$\mathbb{P}\left[X(T)\!=A\right] = \lim_{n\to\infty} \mathbb{P}[X_{T^{[n]}}^{[n])} \geq A]$,
assuming the excess is neglibible in the limit.

- Our previous results for the random walk are

$$\mathbb{P}\big[X^{[n]}_{T^{[n]}} \ge A\big] = P_A \approx \frac{1 - e^{-\theta_n B}}{e^{\theta_n A} - e^{-\theta_n B}}$$

$$\mathbb{E}\big[T^{[n]}\big] \approx \frac{AP_A - B(1 - P_A)}{\mathbb{E}\big[\frac{1}{\sqrt{n}}(Y_1 + \frac{\mu}{\sqrt{n}})\big]}$$

- Show that $\theta_n = -2\mu + o(1)$, i.e., $\lim_n \theta_n = -2\mu$

- Completing this calculation is one way to do Problem 8.15(c) in Ross.

Diffusions as solutions to Stochastic Differential Equations

- The local properties of $\{X(t)\}$ suggest that we can write

$$X(t+h) = X(t) + \mu\big(X(t),t\big)\,h + \sigma\big(X(t),t\big)\big[B(t+h) - B(t)\big] + o(h).$$

- Dividing by h and taking a limit suggests

$$\tfrac{d}{dt}X(t) = \mu\big(X(t),t\big) + \sigma\big(X(t),t\big)\,\tfrac{d}{dt}B(t).$$

- But this derivative is not defined, in the usual sense, since sample paths of $B(t)$ are not differentiable. Why?

- To recognize the lack of differentiability, we write $dX(t) = \mu(X(t), t)\, dt + \sigma(X(t), t)\, dB(t)$. The solution to this **stochastic differential equation** (SDE) corresponds to a **stochastic integral equation**,

$$X(t) = X(0) + \int_0^t \{\mu(X(s), s)ds + \sigma(X(s), s)dB(s)\}.$$

- A stochastic integral can be defined analogously to the Riemannn integral:

$$\int_0^t f(s)\, dB(s) = \lim_{N\to\infty} \sum_{n=0}^{N-1} f\left(\tfrac{tn}{N}\right)\left[B\left(\tfrac{t(n+1)}{N}\right) - B\left(\tfrac{tn}{N}\right)\right].$$

- We note some stochastic calculus results: Supposing $f(s)$ is sufficiently regular and $f(s)$ is independent of $\{B(t), t \geq s\}$,

(i) $\int_0^t f(s)\, dB(s)$ has continuous sample paths.

(ii) $\mathbb{E}\left[\int_0^t f(s)\, dB(s)\right] = 0$. .

(iii) $\text{Var}\left[\int_0^t f(s)\, dB(s)\right] = \mathbb{E}\left[\int_0^t [f(s)]^2\, ds\right]$.

(iv) $\text{Cov}\left[\int_0^t f(s)\, dB(s),\ \int_0^t g(s)\, dB(s)\right]$
$$= \mathbb{E}\left[\int_0^t f(s)\, g(s)\, ds\right].$$

- We can use these properties to check that the solution to
$dX(t) = \mu\big(X(t), t\big)\, dt + \sigma\big(X(t), t\big)\, dB(t)$
is a diffusion with infinitesimal parameters
$\mu(x, t)$ and $\sigma^2(x, t)$.

Solution

Transformations of Diffusions

- If $\{X(t)\}$ is a time-homogeneous diffusion with infinitesimal parameters $\mu_X(x)$ and $\sigma_X^2(x)$, then $Y(t) = f(X(t))$ is also a time-homogeneous diffusion if f is invertible and continous. Why?

- Supposing $f(x)$ is twice differentiable, the infinitesimal parameters of $Y(t)$ are

  $\mu_Y(y) = \mu_X(x) f'(x) + \frac{1}{2}\sigma_X^2(x) f''(x)$

  $\sigma_Y^2(y) = \sigma_X^2(x)[f'(x)]^2$

  where $x = f^{-1}(y)$, $f' = \frac{df}{dx}$, $f'' = \frac{d^2 f}{dx^2}$.

- This transformation formula is different from the non-stochastic "chain rule" for $y = f(x)$, namely $\frac{dy}{dt} = \frac{df}{dx}\frac{dx}{dt}$. In SDE notation,

  $dY(t) = \frac{df}{dx}\, dX(t) + \frac{1}{2}\sigma_X^2 \frac{d^2 f}{dx^2}\, dt.$

Derivation of the transformation formula.

Ornstein-Uhlenbeck process (O-U process)

- If a small particle is suspended in a liquid or gas, then bombardment by molecules of the medium should (by Newton's laws) make the velocity follow a random walk. However, the particule also suffers friction (viscous drag) which is approximately proportional to velocity. This suggests an equation
  $dv(t) = -\alpha v(t)\,dt + \sigma\,dB(t)$
  with location then given by
  $x(t) = x(0) + \int_0^t v(s)\,ds$,
  which is equivalent to
  $dx(t) = v(t)\,dt$.

- The **Ornstein-Uhlenbeck process** is a diffusion on $[-\infty, \infty]$ with infinitesimal parameters $\mu(x,t) = -\alpha x$ and $\sigma^2(x,t) = \sigma^2$.

- Let $X(t) = e^{-\alpha t}X(0) + \int_0^t \sigma e^{-\alpha(t-s)}\,dB(s)$. Assuming that differentiation of stochastic integrals follows the same rules as standard deterministic integrals, show that $X(t)$ is an O-U process.

- This integral shows that the O-U process is a linear function of Brownian motion, and so the O-U process is a Gaussian process. (as long as $X(0)$ has a normal distribution).

- Find the mean and covariance functions for the O-U process using the integral representation and the properties (ii) and (iv) of stochastic integrals. Suppose $X(t)$ is stationary, so that $X(t) = \int_{-\infty}^{t} \sigma e^{-\alpha(t-s)}\, dB(s)$.

- Note: Another way to do this is to use the representation $X(t) = \sigma\, e^{-\alpha t/2}\, B(\alpha e^{\alpha t})$ where $\{B(t)\}$ is standard Brownian motion (see homework).

Exponential Brownian Motion

- Let $Y(t) = \exp\{\sigma B(t) + \mu t\}$. This diffusion model is widely used in modeling the financial markets and population growth.

- Find the infitiesimal parameters for $\{Y(t)\}$ and write down the SDE that it solves.

Diffusion approximation to a branching process

- Let $X(t)$ be a branching process where each individual reproduces at rate λ. Find a diffusion approximation for large values of $X(t)$.