Claim: \( \lim_{m \to \infty} \mathbb{P} \{ \sup_{k > 0} |Z_{m+k} - Z_m| > \epsilon \} = 0 \) implies that \( \{Z_m\} \) is almost surely Cauchy, i.e.

\[
\lim_{m \to \infty} \sup_{k > 0} |Z_{m+k} - Z_m| = 0 \quad \text{with probability 1.}
\]

Proof: Let \( G_m(\epsilon) = \{ \sup_{k > 0} |Z_{m+k} - Z_m| > \epsilon \} \) and \( G(\epsilon) = \bigcap_{m > 0} G_m(\epsilon) \)

\[
= \{ \lim_{m \to \infty} \sup_{k > 0} |Z_{m+k} - Z_m| > \epsilon \}.
\]

Since \( G_m(\epsilon) \) is a decreasing sequence of events for any \( \epsilon > 0 \),

\[
\mathbb{P}[G(\epsilon)] = \mathbb{P}\left[ \lim_{m \to \infty} G_m(\epsilon) \right] = \lim_{m \to \infty} \mathbb{P}[G_m(\epsilon)] = 0.
\]

Now, let \( H_n = \left\{ G\left(\frac{1}{n}\right) \right\}^c \). Then \( H_n \) is decreasing, and

\[
H = \bigcap_{n > 0} H_n = \left\{ \lim_{m \to \infty} \sup_{k > 0} |Z_{m+k} - Z_m| = 0 \right\}.
\]

It follows that

\[
\mathbb{P}(H) = \mathbb{P}\left[ \lim_{n \to \infty} H_n \right] = \lim_{n \to \infty} \mathbb{P}[H_n] = 1.
\]