Answers, Exam II

1. a) Since \( \lim_{x \to \infty} F(x) = c \), we must have \( c = 1 \).
   
   b) Since \( F \) is right continuous,
   
   \[
   F(1) = \lim_{x \uparrow 1} F(x) = \lim_{x \uparrow 1} \frac{1 + x}{2 + x} = \frac{2}{3},
   \]

   and since \( F \) is continuous at \( 1/2 \),

   \[
P(1/2 \leq X \leq 1) = P(X \leq 1) - P(X \leq 1/2) = F(1) - F(1/2) = 2/3 - 1/4 = 5/12.
   \]

2. a) \( p(x) = (2x - 1)/36 \) for \( x = 1, \ldots, 6 \), and \( p(x) = 0 \) otherwise.
   
   b) \( EX = \sum_{x=1}^{6} x p(x) = 161/36 \approx 4.47 \).

3. a) The number of face cards has a hypergeometric distribution with mean \( 5 \times 16/52 \approx 1.538 \).
   
   b) The mass function at \( 2 \) is \( \binom{16}{2} \binom{36}{3} / \binom{52}{5} \approx 33\% \).

4. Let \( X \) denote the number of defects, and let \( B \) be the event that the item was produced on a Monday or Friday. By the law of total probability,

   \[
P(X \leq 1) = P(X \leq 1|B)P(B) + P(X \leq 1|B^c)P(B^c) = 4e^{-3} \cdot \frac{2}{5} + 3e^{-2} \cdot \frac{3}{5} \approx 32.33\%.
   \]

   b) By Bayes law,

   \[
P(B|X = 4) = \frac{P(X = 4|B)P(B)}{P(X = 4|B)P(B) + P(X = 4|B^c)P(B^c)}
   \]

   \[
   = \frac{3^4 e^{-3} (2/5)}{3^4 e^{-3} (2/5) + 2^4 e^{-2} (3/5)} \approx 55.39\%.
   \]

5. a) \( P(X \geq 115) = P(Z \geq -3/4) \approx 77.34\% \).
   
   b) \( 120 - 20(0.25) = 125 \).

6. For \( 0 < y < 1 \),

   \[
   F_Y(y) = P(Y \leq y) = P(1/X^2 \leq y) = P(X \geq 1/\sqrt{y}) = \int_{1/\sqrt{y}}^{\infty} \frac{1}{x^2} \, dx = \sqrt{y}.
   \]

   For \( y \leq 0 \), \( f_Y(y) = 1 \), and for \( y \geq 1 \), \( F_Y(y) = 1 \). The probability density function for \( Y \) is

   \[
f_Y(y) = F'_Y(y) = \begin{cases} 
1/(2\sqrt{y}), & 0 < y < 1; \\
0, & \text{otherwise}.
\end{cases}
   \]