Finding Convex Hull Generators in the Plane

- A set $S \subset \mathbb{R}^d$ is **convex** if $\lambda z_1 + (1 - \lambda)z_2 \in S$ whenever $z_1, z_2 \in S$, and $0 \leq \lambda \leq 1$. In words, $S$ contains all line segments connecting pairs of points in $S$.

- The **convex hull** generated by a set of points $\mathcal{P}$ is the intersection of all convex sets $S$ containing $\mathcal{P}$. If $\mathcal{P} = \{z_i \in \mathbb{R}^d, i = 1 \ldots, n\}$ is finite, this set can be expressed

$$S = \left\{ \sum_{i=1}^{n} \lambda_i z_i \mid 0 \leq \lambda_i \leq 1, \sum \lambda_i = 1 \right\}.$$ 

- If $\mathcal{P}$ is finite, there exists a unique subset $\mathcal{P}^* \subset \mathcal{P}$ of minimal size such that the convex hull of $\mathcal{P}^*$ is identical to the convex hull of $\mathcal{P}$. The set $\mathcal{P}^*$ is called the set of **convex hull generators**.

- Finding the set of convex hull generators of a finite point set $\mathcal{P}$ is a difficult computational problem when the dimension $d$ is greater than 2. If $d = 2$ there are several efficient algorithms.

**Extreme edges algorithm**

- A sufficient condition for $z_1, z_2 \in \mathcal{P}^*$ is that all points in $\mathcal{P}$ are on the same side of the line determined by $z_1$ and $z_2$. Thus if $v$ is perpendicular to $z_1 - z_2$, so $\langle v, z_1 - z_2 \rangle = 0$, then $z_1$ and $z_2$ are in $\mathcal{P}^*$ if and only if for all points $z \in \mathcal{P}$, the inner products $\langle z - z_2, v \rangle$ have the same sign.

- A simple but inefficient algorithm for finding convex hulls in the plane is the **extreme edges algorithm**. Cycle through all $n^2$ pairs $(z_1, z_2)$, and for each pair determine if all the remaining points lie on the same side of the line determined by $(z_1, z_2)$. This algorithm has complexity $n^3$. 


Jarvis march algorithm

- The Jarvis march algorithm, sometimes called the gift wrapping algorithm, visits the points on the convex hull in order (unlike the extreme edges algorithm).

1. Start with any point on the hull. The point with greatest x-coordinate is a natural choice. Call this point \((X_0, Y_0)\).
2. Scan through all points \((X_i, Y_i)\) and locate the point such that the angle from \((1, 0)\) to \((X_i - X_0, Y_i - Y_0)\) is least. This is the next counterclockwise point from \((X_0, Y_0)\) on the hull, call it \((X_1, Y_1)\).
3. Suppose we have located points \((X_i, Y_i)\), \(i = 1, \ldots, m\) that occur in counterclockwise order on the hull, where \(m \geq 2\). Compute all angles between the vectors \((X_i - X_m, Y_i - Y_m)\) and \((X_{m-1} - X_m, Y_{m-1} - Y_m)\), and find the point \(i\) that gives the smallest positive angle. Add this point to the hull.
4. Return to 3 until \((X_m, Y_m) = (X_0, Y_0)\).
To compute the angle $\theta$ counter-clockwise from unit vector $U$ to unit vector $V$: (i) let $D$ denote the dot product $\langle U, V \rangle$, (ii) let $C$ denote the cross product $U_x V_y - U_y V_x$, (iii) if $C \geq 0$, $\theta = \text{acos}(D)$, otherwise $\theta = 2\pi - \text{acos}(D)$.

The Jarvis march algorithm has worst-case complexity $n^2$, which occurs if every point lies on the hull. In general, if $h$ points lie on the hull then the complexity is $nh$.

Quickhull

Among the points in $\mathcal{P}$, the four cardinal points are defined as follows:

- $(X_1, Y_1), \quad X_1 = \max\{X_i | (X_i, Y_i) \in \mathcal{P}\}$
- $(X_2, Y_2), \quad Y_2 = \max\{Y_i | (X_i, Y_i) \in \mathcal{P}\}$
- $(X_3, Y_3), \quad X_3 = \min\{X_i | (X_i, Y_i) \in \mathcal{P}\}$
- $(X_4, Y_4), \quad Y_4 = \min\{Y_i | (X_i, Y_i) \in \mathcal{P}\}$

Without loss of generality assume that the origin is contained in the interior of $\mathcal{P}$. The transform $X_i \rightarrow X_i - \bar{X}, \ Y_i \rightarrow Y_i - \bar{Y}$ will ensure this. The four cardinal points partition $\mathcal{P}$ into four sectors.

For a given sector (say the sector determined by $(X_1, Y_1)$ and $(X_2, Y_2)$), construct the line $\mathcal{L}$ determined by the sector boundaries (i.e. the line from $(X_1, Y_1)$ to $(X_2, Y_2)$).

For each point $(X_k, Y_k)$ in the sector, calculate the perpendicular distance $D_k$ from $(X_k, Y_k)$ to $\mathcal{L}$. Let $(X_q, Y_q)$ be the point in the sector that is furthest from the line, subject to being on the opposite side of the origin relative to the line. This point must also be a convex hull generator.

The new point $(X_q, Y_q)$ divides the original sector into two sectors. Call the previous step recursively on these two sectors (i.e. on the sector determined by $(X_1, Y_1)$ and $(X_q, Y_q)$ and on the sector determined by $(X_q, Y_q)$ and $(X_2, Y_2)$).
• The complexity is order $n \log n$ as long as the sectors are divided roughly in half at each step.

**Graham’s Scan**

• Graham’s Scan is a worst-case $n \log(n)$ algorithm for finding the convex hull of a set of points in the plane.

• The key idea is that for any three consecutive convex hull generators, the included angle will measure less than $\pi$ radians.

1. Let $P_0$ be the point with greatest $y$ coordinate.
2. Sort the other points in increasing angle relative to an origin at $P_0$. Let $P_1, \ldots, P_{N-1}$ denote this list. Let $Q_1 = P_0$, $Q_2 = P_1$, and $Q_3 = P_2$.
3. Let $\alpha$ be the counterclockwise angle from $Q_2 \to Q_3$ to $Q_2 \to Q_1$.
4. If $\alpha \leq \pi$ set $Q_1 = Q_2$, set $Q_2 = Q_3$, and set $Q_3$ to the next point after $Q_2$.
5. If $\alpha > \pi$, delete $Q_2$ from the list, set $Q_2 = Q_1$, and set $Q_1$ to be the first point before $Q_2$ that has not been deleted.
6. If $Q_3 = P_0$ stop. Otherwise return to 3.

• **Complexity:** $Q_3$ never moves backward, so there can be at most $n$ repetitions of step 4. There can be at most $n$ repetitions of step 5 since there are only $n$ points to delete. Thus there is only linear work to do after the sort, so the sort dominates at complexity $n \log n$.

• A proof that Graham’s scan actually computes the convex hull relies on two points:

  1. Every deleted point is not on the convex hull.
  2. Whenever step 4 is successfully completed, the set of non-deleted points from $P_0$ up to $Q_2$ are the vertices of a convex polygon.

To prove point 1, note that if $Q_2$ is to be deleted, it lies in the interior of the triangle formed by $P_0, Q_1$ and $Q_3$. Therefore it cannot be a hull generator.

Point 2 follows by induction. At the start, the set of points is $P_0, P_1, P_2$, and any triangle is a convex polygon. If at some stage a point is deleted, the property of being a convex polygon is preserved. If a new point is added, since the angle of the new point with $P_0$ is greater than the angle for any other point, a new convex polygon is obtained.

To conclude, point 1 implies that the final set of points is a superset of the convex hull generators. But if it contains any points not on the hull, then the set would not form a convex polygon.