CONSISTENT ESTIMATES OF THE PARAMETERS OF A LINEAR SYSTEM

BY WILLIAM N. ANDERSON, JR., GEORGE B. KLEINDORFER, PAUL R. KLEINDORFER, AND MICHAEL B. WOODROOFE

Carnegie-Mellon University

1. Introduction. We will be concerned with the following dynamic linear system which finds application in both economics and engineering, for example Aoki [3] and Griliches [6] have used this model.

\[
\begin{align*}
    x_{k+1} &= Ax_k + u_k, & k \geq 0 \\
    y_k &= x_k + w_k, & k \geq 1.
\end{align*}
\]

In (1.1), the state equation, \( x_k \) is a \( p \)-dimensional column vector which represents the state of some system at time \( k \); \( A \) is a \( p \times p \) transition matrix; and \( u_k \) represents a random disturbance, or noise.

In (1.2), the observation equation, \( y_k \) represents an observation made on the system at time \( k \), and \( w_k \) represents noise. We will assume that \( u_0, u_1, \cdots \) and \( w_1, w_2, \cdots \) are independent sequences of zero mean, independent and identically distributed random vectors with covariance matrices \( V \) and \( W \) respectively and that \( x_0 \) is independent of the \( u_i \)'s and \( w_i \)'s and has finite covariance matrix. We remark, in passing, that the superficially more general model in which (1.2) is replaced by

\[
y_k = Mx_k + v_k, \quad k \geq 1,
\]

where \( M \) is nonsingular, may be reduced to (1.2) by an appropriate change of bases.

When \( A, V, W \) and the distribution of \( x_0 \) are known, linear least squares prediction and filtering may be done with the Kalman Filter [10], which provides a method for computing the projections, \( x_{1|k} \) and \( y_{1|k} \), of \( x_i \) and \( y_i \) on the Hilbert subspace spanned by \( y_1, \cdots, y_k \). Specifically,

\[
\begin{align*}
    x_{k|k} &= (I - \Delta_k)Ax_{k-1|k-1} + \Delta_k y_k, & k \geq 1, \\
    x_{1|k} &= A^{t-k}x_{k|k}, \\
    y_{1|k} &= x_{1|k}, \quad t > k,
\end{align*}
\]

where \( I \) denotes the \( p \times p \) identity matrix and \( x_0|0 = E[x_0] \). The matrix \( \Delta_k \) appearing in (1.3) is determined by

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(1.4a) \[ S_k = AP_{k-1}A' + V \]

(1.4b) \[ \Delta_k = S_k(S_k + W)^+ \]

(1.4c) \[ P_k = (I - \Delta_k)S_k, \quad k \geq 1, \]

where $^+$ denotes pseudo-inverse, $'$ denotes transpose, and $P_0$ is the covariance matrix of $x_0$.

In practice, however, $A$, $V$, and $W$ will often be unknown, so that two problems arise in connection with the Kalman Filter. First, the parameters $A$, $V$, and $W$ must be estimated from the $y_k$'s; and second, the effects of replacing $A$, $V$, and $W$ by estimates in (1.4) should be considered. In this paper we will present estimates of $A$, $V$, and $W$, and show that they are strongly consistent when the system (1.1) is stable, that is when $\rho(A)$, the spectral radius of $A$, is less than one. We will then determine the asymptotic behavior as $k \to \infty$ of (1.4) and show that it is unchanged if $A$, $V$, and $W$ are replaced by strongly consistent estimates. Our results are stated precisely in Section 2 and proved in Sections 3, 4 and 5. Theorem 2.3 and Section 4 are independent of the remainder of the paper.

Other approaches to the problem of parameter estimation in (1.1) and (1.2) and/or determining the effect of replacing $A$, $V$, and $W$ by estimates in the Kalman Filter may be found in [3], [4], [7], [9], and [13]. These authors, however, have not been primarily concerned with analytical results; in fact, only [2] and [5] even consider the consistency of their estimates. Somewhat more theoretical work has been done on parameter estimation in linear-stochastic difference equations with independent inputs, of which (1.1) and (1.2) are a special case if $W = 0$ ([16], [17], and [19]). The presence of a non-zero $W$ in (1.2), however, introduces major complications in the filtering and prediction problems (taking $W = 0$ in (1.4) yields $P_k = 0$, $S_k = V$, and $\Delta_k = VV'$, $k \geq 1$) as well as some complications in the parameter estimation problem. The main results of Section 4 on the asymptotic behavior of (1.4) have been proved by Kalman and Bucy [12] for the continuous case; i.e. when (1.1) is a differential instead of a difference equation. Theorem 2.3 has been proven via Lyapunov theory by Kalman ([11], page 371) under somewhat stronger conditions. However, the proof is not given explicitly for the discrete case, so we have included a proof here by other means.

2. Statement of the theorems. In order to state our results precisely, we will need the following notation. We will denote by $R^p$ and $\mathcal{S}^p$ respectively the real linear spaces of $p$-dimensional column vectors with real components and $p \times p$ matrices with real entries. The topologies in $R^p$ and $\mathcal{S}^p$ will be determined by the Euclidian norms.

\[ |x| = (x'x)^{1/2}, \quad x \in R^p \]

\[ \|G\| = [\text{tr} (GG')]^{1/2}, \quad G \in \mathcal{S}^p. \]

If $G \in \mathcal{S}^p$ is symmetric, then $G > 0$ and $G \succeq 0$ mean that $G$ is positive definite (pd) and positive semi-definite (psd) respectively, and if $F$, $G \in \mathcal{S}^p$ are symmetric,
then $F \succeq G$ iff $F - G \succeq 0$. Finally, we will need the notion of parallel addition which is defined for psd matrices $F, G$ by

$$F : G = F (F + G)^{+} G.$$  

$F : G$ is called the parallel sum of $F$ and $G$ and is studied in detail by Anderson and Duffin [1], [2].

We will estimate the parameter $A$ of (1.1) by

$$(2.1)
\hat{A}_n = (\sum_{k=0}^{n} y_k y_k') (\sum_{k=0}^{n} y_{k-1} y_{k-1}')^+, \quad n \geq 3.
$$

The estimate $\hat{A}_n$ is suggested by the fact that $E[y_k y_k'] = A E[y_{k-1} y_{k-1}]$, $k \geq 3$, and enjoys the following consistency property.

**Theorem 2.1.** If $\rho(A) < 1$, and if $A$ and $V$ are nonsingular, then $\hat{A}_n$ is a strongly consistent estimate of $A$, that is, $\hat{A}_n \rightarrow A$ with probability one as $n \rightarrow \infty$.

Theorem 2.1 will be proved in Section 3. Granting its validity for the moment, we may then estimate $V$ and $W$ as follows. Define

$$B_1 = E[(y_k - A y_{k-1})(y_k - A y_{k-1})']
= V + W + AWA',$$

$$B_2 = E[(y_k - A y_{k-1})(y_k - A y_{k-1})']
= V + W + AVA' + A^2 WA';$$

then, if $A$ is nonsingular, $B_1, B_2$, and $A$ uniquely determine $V$ and $W$ by

$$W = \frac{1}{2} \{B_1 + A^{-1} (B_1 - B_2) A^{-1}\},$$

$$V = B_1 - W - AWA'.$$

Therefore, strongly consistent estimates of $A, B_1$, and $B_2$ determine strongly consistent estimates of $V$ and $W$.

**Theorem 2.2.** If $\hat{A}_n$ is any strongly consistent estimate of $A$ and if $\rho(A) < 1$, then

$$(2.2)
B_{n,i} = 1/n \sum_{k=0}^{n} (y_k - \hat{A}_k y_{k-i})(y_k - \hat{A}_k y_{k-i})', \quad n \geq 3,
$$

is a strongly consistent estimate of $B_i, i = 1, 2$. In particular if $A$ and $V$ are nonsingular, and if $\hat{A}_n$ is given by (2.1) then $B_{n,i}$ is a strongly consistent estimate of $B_i$.

**Remark.** We have used $\hat{A}_n$ rather than $\hat{A}_n$ in (2.2) in order to make the computation of $B_{n,i}$ Markovian. It will be clear from the proof of Theorem 2.2 however, that $B_{n,i}$ would still be strongly consistent if $\hat{A}_n$ were replaced by $\hat{A}_n$ in (2.2).

Given any strongly consistent estimates $\hat{A}_n, \hat{V}_n$, and $\hat{W}_n$ of $A, V$, and $W$ respectively it is natural to approximate the Kalman Filter by

$$\hat{S}_k = \hat{A}_k \hat{P}_{k-1} \hat{A}_k' + \hat{V}_k$$

$$(2.3)
\Delta_k = \hat{S}_k (\hat{S}_k + \hat{W}_k)^{+}
\hat{P}_k = (I - \hat{A}_k) \hat{S}_k.$$
(2.4) \[ \hat{x}_k |_k = (1 - \hat{\Delta}) \hat{A}_k \hat{x}_{k-1} |_{k-1} + \hat{\Delta}_k y_k, \quad k \geq 1, \]

where \( I \) denotes the \( p \times p \) identity matrix and \( P_0 \) may be any pd matrix. A natural object of interest is then the asymptotic behavior of \( \hat{\Delta}_k \rightarrow \Delta_0 \) and \( \hat{x}_k |_k \rightarrow \hat{x}_k |_k \) as \( k \rightarrow \infty \). Our analysis of this behavior requires knowledge of the asymptotic behavior of \( S_k \) as \( k \rightarrow \infty \), which, of course, is of interest in its own right.

**Theorem 2.3.** Let \( V > 0 \) and let \( W \geq 0 \); define \( \phi \) on the set \( S \) of pd matrices by

(2.5) \[ \phi(S) = A(S:W)A' + V, \quad S \in S. \]

Then \( S_k = \phi(S_{k-1}), k \geq 2 \). Moreover, \( \phi \) has a unique positive definite fixed point \( S_\circ \), and \( \phi^n(S) \rightarrow S_\circ \) uniformly on \( S \) as \( n \rightarrow \infty \), where \( \phi^n \) denotes the \( n \)-th iterate of \( \phi \).

**Corollary 2.1.** Let \( V > 0 \), then \( S_k \rightarrow S_\circ \), \( \Delta_k \rightarrow \Delta_0 = S_0(S_0 + W)^{-1} \), and \( P_k \rightarrow P_0 = S_0 - S_0(S_0 + W)^{-1}S_0 \) as \( k \rightarrow \infty \).

**Corollary 2.2.** For \( A \in \mathbb{R}^p \), \( W \geq 0 \), and \( V > 0 \), define \( S_\circ(A, V, W) \) to be the unique positive definite fixed point of the function \( \phi \) defined by (2.5); then \( S_\circ(A, V, W) \) depends continuously on \( (A, V, W) \).

**Remark.** Theorem 2.3 and its corollaries have applications in the study of asymptotic properties of certain classes of optimal control problems via the duality theorem of Kalman [10].

The proofs of Theorem 2.3 and Corollary 2.2 will be presented in Section 4 together with an example illustrating some difficulties which may arise if \( V \) is not pd. Corollary 2.1 is an obvious consequence of Theorem 2.3. We now consider the asymptotic behavior of \( \hat{S}_k \) and \( \hat{\Delta}_k |_k \). Theorems 2.4 and 2.5 (below) will be proved in Section 5; their corollaries are obvious.

**Theorem 2.4.** If \( V > 0 \), and if \( \hat{A}_n, \hat{V}_n, \) and \( \hat{W}_n \) are strongly consistent estimates of \( A, V, \) and \( W \) for which \( \hat{V}_n > 0 \) and \( \hat{W}_n \geq 0 \) for all \( n \geq 1 \), then \( \hat{S}_k \rightarrow S_\circ \) with probability one as \( k \rightarrow \infty \).

**Corollary 2.3.** If the hypotheses of Theorem 2.4 are satisfied, then \( \hat{\Delta}_k \rightarrow \Delta_0 \) with probability one as \( k \rightarrow \infty \).

**Theorem 2.5.** If \( V > 0 \), if \( \rho(A) < 1 \), and if \( \hat{A}_n, \hat{V}_n, \hat{W}_n \) are strongly consistent estimates of \( A, V, \) and \( W \) for which \( \hat{V}_n > 0 \) and \( \hat{W}_n \geq 0 \) for all \( n \geq 1 \), then

\[ \lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^{n} |\hat{x}_k |_k - \hat{x}_k |_k | = 0 \quad \text{with probability one}. \]

**Corollary 2.4.** If the hypotheses of Theorem 2.5 are satisfied, then

\[ \lim_{n \rightarrow \infty} n^{-1} \sum_{k=1}^{n} |\hat{x}_{k+r} |_k - \hat{A}_k \hat{x}_k |_k | = 0 \quad \text{with probability one}. \]

for any \( r \geq 1 \).

**3. Consistency of the estimates.** In this section we will establish Theorems 2.1 and 2.2; accordingly we assume throughout that \( \rho(A) < 1 \). Define

\[ z_k = w_k + v_k - A w_{k-1} = y_k - A y_{k-1}, \quad k \geq 2 \]

\[ R_n = (\sum_{k=1}^{n} z_k y_{k-1})(\sum_{k=1}^{n} y_{k-1} y_{k-1})^+ \quad n \geq 3. \]
To prove Theorem 2.1, we will show that, with probability one,

\begin{align}
(3.1) & \quad n^{-1} \sum_{k=1}^n z_k y_{k-1} \to 0 \\
(3.2) & \quad n^{-1} \sum_{k=0}^n y_k y_{k-2} \to \psi
\end{align}

where \( \psi \) is nonsingular. It then follows that for large \( n \) the left side of (3.2) is nonsingular, and thus for large \( n \), by a simple computation, that \( \hat{A}_n = A + R_n \). Thus, (3.1) and (3.2) suffice to prove Theorem 2.1.

To establish (3.1) we need first a bound on the covariance matrix of \( y_k \). We have from (1.1) and (1.2)

\begin{equation}
(3.3) \quad y_k = w_k + \sum_{j=0}^{k-1} A^j v_{k-1-j} + A^k x_0
\end{equation}

from which it follows that

\begin{align}
(3.4a) & \quad E(y_k) = A^k E(x_0) \\
(3.4b) & \quad \Sigma_k = \text{Cov} (y_k) = W + \sum_{j=0}^{k-1} A^j V A^j + A^k \text{Cov} (x_0) A^k
\end{align}

\( \to W + \sum_{j=0}^{\infty} A^j V A^j = \Sigma \text{ say}, \)

as \( k \to \infty \). Here we have used the fact that \( \lim_{n \to \infty} \|A^n\| n^{-1} = \rho(A) < 1 \) ([15], page 75). Define

\begin{align}
S_{n,1} &= \sum_{k=0}^{n-1} k^{-1} w_k y_{k-2} \\
S_{n,2} &= \sum_{k=0}^{n-1} k^{-1} (v_{k-1} - A w_{k-1}) y_{k-1}, \quad n \geq 3,
\end{align}

and for \( n \geq 3 \) let \( \varpi_n \) be the smallest \( \sigma \) algebra with respect to which \( x_0, v_0, \cdots, v_n, \)

\( w_1, \cdots, w_n \) are measurable. If \( a, b \in B^p \) it is easily seen that \( \{a' S_{n,1} b; \varpi_n; n \geq 3\} \)

and \( \{a' S_{n,2} b; \varpi_{n-1}; n \geq 3\} \) are martingales, \( i = 1, 2 \); for example

\begin{align}
E(a' S_{n+1,1} b | \varpi_n) - a' S_n b &= (n + 1)^{-1} a' E(w_{n+1} y_{n-1} | \varpi_n) b \\
&= (n + 1)^{-1} a' E(w_{n+1}) y_{n-1} b = 0.
\end{align}

Moreover, by the mutual independence of \( w_1, w_2, \cdots, \) and the independence of \( w_0, w_{n+1}, \cdots, \) from \( y_1, \cdots, y_{k-1} \), we find

\begin{align}
E\{a' S_n b \}^2 &= \sum_{k=0}^n k^{-2} E\{(a' w_k y_{k-2})^2\} \\
&= \sum_{k=0}^n k^{-2} (a' W a) b' E[y_{k-2} y_{k-2}] b
\end{align}

and (similarly)

\begin{align}
E\{a' S_n b \}^2 &= \sum_{k=0}^n k^{-2} [a' (V + AWA') a] b' E[y_{k-2} y_{k-2}] b
\end{align}

are bounded for \( n \geq 3 \) by (3.4). The martingale convergence theorem ([5], page 319) therefore asserts that \( \lim_{n \to \infty} a' S_n b \) exists and is finite with probability
one, \( i = 1, 2 \). It now follows from the Kronecker Lemma (\cite{14}, page 238) (which asserts that if \( \sum a_n \) converges and \( b_n \to \infty \), then \( b_n^{-1} \sum_{i=1}^{n} b_i a_i \to 0 \) that

\[
a'(n^{-1} \sum_{k=0}^{n} v_{k-1} y_{k-1}) b \to 0
\]

\[
a'(n^{-1} \sum_{k=0}^{n} (v_{k-1} - A v_{k-1}) y_{k-2}) b \to 0
\]

with probability one. Equation (3.1) now follows by the arbitrariness of \( a \) and \( b \).

To establish (3.2) we will take advantage of the fact that \( y_k \) in (3.3) is almost a moving average of the \( v_i \)'s and \( w_i \)'s. Let \( v_{n-1} v_{n-2} \cdots \) be a sequence of independent random vectors which have the same distribution as the \( v_i \)'s and are mutually independent of \( x_0 \), \( v_0 \), \( v_1 \), \ldots, and \( v_k \), \( w_0 \), \ldots; such a sequence may always be found by possibly enlarging the probability space (\cite{5}, page 71). We now define random vectors \( u_k \) and \( q_k \) as

\[
y_k = u_k - q_k ,
\]

\[
u_k = w_k + \sum_{j=0}^{n} A^j v_{k-j-1} ,
k \geq 1 ,
\]

\[
q_k = A^k (\sum_{j=0}^{n} A^j v_{k-j-1} - x_0) = A^k q_k ,
k \geq 0 .
\]

Using \( \rho(A) < 1 \), it may be shown by the Three Series Theorem (\cite{3}, page 111) that \( u_k \) and \( q_k \) are well-defined random vectors. Here \( u_k \), \( k \geq 1 \), is a moving average of the \( v_i \)'s and \( w_i \)'s and, therefore, a metrically transitive, strictly stationary process (\cite{5}, page 460); and \( |q_k| \to 0 \) with probability one and mean square as \( k \to \infty \). Equation (3.2) is now a special case (\( i = 1 \)) of the following lemmas.

**Lemma 3.1.** Let \( i \geq 0 \) be an integer. Then

\[
E[|u_{k} u'_{k-i}|] \leq E[|u_{k}|^2] = \text{tr} (\varPhi) < \infty ;
\]

(ii) \( E[u_k u_{k-i}] = A^i (\varPhi - W) + \delta_{i,0} W \)

where \( \delta_{i,j} \) is the Kronecker \( \delta \). If \( A \) and \( V \) are nonsingular, then so is \( E[u_k u'_{k-i}] \).

**Lemma 3.2.** Let \( i \geq 0 \) be an integer; then

\[
\text{lim}_{n \to \infty} n^{-1} \sum_{k=0}^{n} y_{k} y'_{k-i} = \text{lim}_{n \to \infty} n^{-1} \sum_{k=0}^{n} u_{k} u'_{k-i} = E[u_k u'_{k-i}]
\]

\[
\text{lim}_{n \to \infty} n^{-1} \sum_{k=0}^{n} ||y_{k} y'_{k-i}|| = \text{lim}_{n \to \infty} n^{-1} \sum_{k=0}^{n} ||u_{k} u'_{k-i}|| = E[||u_k u'_{k-i}||].
\]

**Proof.** Equation (ii) of Lemma 3.1 follows from (3.3) and (3.4) by a routine computation and the remark that \( \varPhi - W \) is pd if \( V \) is nonsingular. Thereafter, (i) follows from

\[
E[||u_{k} u'_{k-i}||] = E[|u_{k}|^2] \leq E[|u_{k}|^2] = E[\text{tr} (u_k u'_{k})] = \text{tr} (\varPhi) .
\]

The final equalities in (3.8a) and (3.8b) follow from the ergodic theorem and Lemma 3.1, since \( u_k u'_{k-i}, k \geq i \), is again a metrically transitive, strictly stationary process. Therefore, Lemma 3.2 would follow from
(3.9) \[ u_{k-1} - y_{k-1} = u_{k-1} + q u_{k-1} + q y_{k-1} \to 0 \]

with probability one as \( k \to \infty \). Since \( \|A^k\| \to 0 \) exponentially fast as \( k \to \infty \), (3.9) follows from (3.7) and the fact that \( \sup_{\|x\| \leq 1} \|Ax\| \leq \sup_{\|x\| \leq 1} 1 - \|u\| \)

\(< \infty \) with probability one. □

To establish Theorem 2.2, define

\[ B_{n,i} = n^{-1} \sum_{k=0}^n (y_k - A^i y_{k-i})(y_k - A^i y_{k-i})' \]

then by the ergodic theorem \( B_{n,i} \to B_i \) with probability one as \( n \to \infty \) because the sequences \( (y_k - A^i y_{k-i})(y_k - A^i y_{k-i})' \) are strictly stationary and \((i+1)\)-dependent and therefore metrically transitive. Therefore, it will be sufficient to show that

\[ B_{n,i} - B_{n,i} = n^{-1} \sum_{k=0}^n (A - \hat{A}) y_{k-i}(A - \hat{A}) + n^{-1} \sum_{k=0}^n (A - \hat{A}) y_{k-i} y_{k-i}' \]

\[ + n^{-1} \sum_{k=0}^n (A - \hat{A}) y_{k-i} y_{k-i} A' \]

(3.10)

\[ + n^{-1} \sum_{k=0}^n (A - \hat{A}) y_{k-i} y_{k-i}' \]

converges to zero as \( n \to \infty \), \( i = 0, 1, 2 \). This, however, is an easy consequence of Lemma 3.2. For example, it follows from Lemma 3.2 that for any \( m \geq 3 \)

\[ \lim_{n \to \infty} \sup \| n^{-1} \sum_{k=0}^n y_{k-i} y_{k-i}' (A - \hat{A}) \| \]

\[ \leq \sup_{k \geq m} \| A - \hat{A} \| \lim_{n \to \infty} \sup n^{-1} \sum_{k=m}^n \| y_{k-i} \| \]

\[ = \sup_{k \geq m} \| A - \hat{A} \| \| E \| \| u_{k-i} \| \]

which may be made arbitrarily small by proper choice of \( m \). The other sums in (3.10) may be handled similarly, thus completing the proof of Theorem 2.2.

4. Asymptotic behavior of \( S_n \). In this section we will prove Theorem 2.3, which asserts the existence of a unique fixed point for the \( \phi \) defined by \( \phi(S) = A(S;W)A' + V \), the set of pd matrices. For this purpose we will obviously need to know some properties of the parallel sum \( (S;W) = S + W \) of \( W \). Since we consider parallel addition only when one of the summands is pd, the pseudo-inverse appearing in its definition is really a true inverse. This fact simplifies the proof of the following lemmas considerably (cf. [2]).

Lemma 4.1. Let \( G \) be the set of \( (F, G) \in \mathbb{R}^p \times \mathbb{R}^p \) for which \( F \geq 0, G \geq 0 \), and \( F + G > 0 \); then

(i) parallel addition is continuous when restricted to \( G \);

(ii) \( F + G = G + F \) if \( F, G \in G \);

(iii) if \( (F, G) \in G \) and \( F \leq H \), then \( (F, G) \leq (H, G) \); and

(iv) \( F + G = F + (F + G)^{-1}F \leq F, (F, G) \in G \).

Proof. (i) is obvious since matrix inversion and multiplication are continuous operations. In the special case that \( F \) and \( G \) are pd, (ii) and (iii) are also obvious from \( (F, G)^{-1} = F^{-1} + G^{-1} \); and the general case follows from the special one by considering \( F = F + \epsilon I \) and \( G = G + \epsilon I \) as \( \epsilon \to 0 \). Finally, (iv) follows from

\[ F + G = F + (F + G)^{-1}(F + G - F) = F + (F + G)^{-1}F. \]
We will also need the following lemma, which is the easy half of a theorem due to Stein (see [8]).

**Lemma 4.2.** Let $D \in \mathfrak{R}^n$. If there exists a pd matrix $F$ for which $F - D'FD$ is pd, then $\rho(D) < 1$.

**Proof.** Let $F$ be such a matrix and let $\lambda$ be any (possibly complex) eigenvalue of $D$; then there is an $x \in \mathbb{R}^n$ for which $Dx = \lambda x$ and, consequently,

$$x'(F - D'FD)x = (1 - |\lambda|^2)x'Fx > 0.$$ 

It follows that $|\lambda| < 1$ and, therefore, that $\rho(D) < 1$. $\square$

The first step in the proof of Theorem 2.3 will be to verify that if $V > 0$, then $S_k = \phi(S_{k-1})$, $k \geq 2$. If $S_{k-1}$ is pd, then from (1.4) and Lemma 4.1 (iv)

$$S_k = A[S_{k-1} - S_{k-1}(S_{k-1} + W)^{-1}S_{k-1}]A' + V$$

$$= A(S_{k-1};W)A' + V = \phi(S_{k-1}),$$

which is again pd by Lemma 4.1 (ii). Therefore, since $S_1$ is pd by (1.4), (4.1) must hold for $k \geq 2$.

Next we show that $\phi$ has at most one fixed point. Toward this end we observe that if $T_1$ and $T_2$ are any two fixed points of $\phi$, then by parts (ii) and (iv) of Lemma 4.1

$$T_1 - T_2 = A[(T_1;W) - (T_2;W)]A' = AW[(T_1 + W)^{-1} - (T_2 + W)^{-1}]WA'$$

$$= AW(T_1 + W)^{-1}T_1 - T_2(T_2 + W)^{-1}WA' = D_1(T_1 - T_2)D_2^{-1},$$

where $D_i = AW(T_i + W)^{-1}$; $i = 1, 2$. Therefore, it will suffice to show that $\rho(D_i) < 1$, $i = 1, 2$. For later reference we state this fact as

**Lemma 4.3.** Let $V$ be pd; let $T$ be any fixed point of $\phi$, and let

$$D = AW(T + W)^{-1};$$

then $\rho(D) < 1$.

**Proof.** Since $\rho(D) = \rho(D')$ it will suffice by Lemma 4.2 to exhibit a pd matrix $F$ for which $F - D'FD$ is pd; but $F = T$ is such a matrix, for

$$T - DTD' = AW(T + W)^{-1}TA' + V - AW(T + W)^{-1}T(T + W)^{-1}WA'$$

$$= AW(T + W)^{-1}T[I - (T + W)^{-1}W]A' + V$$

$$= AW(T + W)^{-1}T(T + W)^{-1}TA' + V$$

$$= AT(T + W)^{-1}W(T + W)^{-1}TA' + V. \square$$

To complete the proof of Theorem 2.3, we observe first that for any $S \in \mathfrak{S},$

$$V \leq \phi(S) = A(S;W)A' + V = A[W - W(S + W)^{-1}W]A' + V$$

$$\leq AW'A' + V$$
by Lemma 4.1 (iv). In particular, $V \preceq \phi(V)$, from which it follows by induction from Lemma 4.1 (iii) that

$$
\phi^n(V) \preceq \phi^{n+1}(V) \preceq AW_1 + V,
$$

$n \geq 1$.

Therefore, $\lim_{n \to \infty} \phi^n(V) = S_0$ exists, ([18] page 263) and since $\phi$ is continuous on $S$, $S_0$ must be a fixed point. Similarly, $\lim_{n \to \infty} \phi^n(AWA' + V)$ is a fixed point which must, therefore, equal $S_0$. The uniformity statement in Theorem 2.3 now follows from (4.2); indeed,

$$
\phi^n(V) \preceq \phi^{n+1}(S) \preceq \phi^n(AWA' + V)
$$

for all $S \in S$ and $n \geq 1$. [2]

To establish Corollary 2.2 let $A_n \to A, V_n \to V > 0, and W_n \to W$ with $V_n > 0$ and $W_n \geq 0$ for all $n \geq 1$; then, setting $S_0^n = S_0(A_n, V_n, W_n), n \geq 1$,

$$
S_0^n = A_n(S_0^n; W_n)A_n' + V_n \preceq A_nW_nA_n' + V_n,
$$

$n \geq 1$, by (4.2). Therefore, $S_0^n$ is bounded. Moreover, if $S$ is any limit point of $S_0$, then $S = A(S; W)A' + V$ by Lemma 4.1 (i). Therefore, $S_0(A, V, W)$ is the unique limit point of $S_0^n$. [2]

Finally, we remark that if (2.5) were used to define $\phi$ on the set of all psd matrices, and if the requirement that $V$ be pd were dropped, then the extended $\phi$ need not have a unique fixed point. For example, let $A = 2I$,

$$
V = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{ and } W = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix};
$$

then it is easily verified that $V$ and $V + 3W$ are both solutions of the equation $S = 4(S; W) + V$.

5. Asymptotic behavior of $\hat{S}_k$ and $\hat{x}_{h+k}$. In this section we will prove Theorems 2.4 and 2.5, which compare the asymptotic behaviors of $\hat{S}_k$ and $\hat{x}_{h+k}$ with those of $S_0$ and $S_{h+k}$ respectively. To establish Theorem 2.4 it will clearly suffice to show that if $\hat{A}_n, \hat{V}_n$, and $\hat{W}_n$ are any fixed sequences of matrices for which $\hat{A}_n \to \hat{A}, \hat{V}_n \to V > 0$ and $\hat{W}_n \to W \geq 0$, with $\hat{V}_n > 0$ and $\hat{W}_n \geq 0$ for all $n \geq 1$, then $\hat{S}_k \to S_0$, where $\hat{S}_k$ is defined by (2.3) and $S_0$ is as in Theorem 2.3. Let $\hat{A}_n, \hat{V}_n$, and $\hat{W}_n$ be such sequences and define $\phi_n, n \geq 1$ by

$$
\phi_n(S) = \hat{A}_n(S; \hat{W}_{n-1})\hat{A}_n' + \hat{V}_n, \quad S \in S;
$$

then by (4.3) there is a compact subset $S_0 \subseteq S$ for which $\phi_n(S) \subseteq S_0$ for $n \geq 1$, and by Lemma 4.1 $\phi_n \to \phi$ uniformly on $S_0$. We now observe that the estimate $\hat{S}_n$ of $S_n$ may be written

$$
\hat{S}_n = \phi_n \circ \cdots \circ \phi_k(\hat{S}_1), \quad n \geq 1,
$$

where $\circ$ denotes composition, $\hat{S}_1 > 0$ by (2.3), and $\hat{S}_k \in S_0, k \geq 2$. Let $\epsilon > 0$; then by Theorem 2.3 there is an integer $r = r_\epsilon$ for which $\|S_0 - \phi'(S)\| \leq \epsilon$,
for all \( S \in \mathcal{S} \); and since \( \mathcal{S}_0 \) is compact, we may select a subsequence \( k_i, i \geq 1 \), for which

\[
\lim_{i \to \infty} \|\hat{S}_{k_i} - S_0\| = \lim_{i \to \infty} \sup \|\hat{S}_k - S_0\|, \quad \lim_{i \to \infty} \hat{S}_{k_{i-r}} = T \in \mathcal{S}_0.
\]

By the uniform convergence of \( \phi, \phi \) on \( \mathcal{S}_0 \), we must then have \( \lim_{i \to \infty} \hat{S}_k = \phi(T) \) and, therefore,

\[
\lim_{i \to \infty} \|\hat{S}_k - S_0\| = \lim_{i \to \infty} \|\hat{S}_{k_i} - S_0\| = \|\phi(T) - S_0\| \leq \epsilon.
\]

Since \( \epsilon \) is arbitrary, Theorem 2.4 follows. 

Finally, to prove Theorem 2.5 we write

\[
\begin{align*}
\tilde{x}_{k_{j+1}} &= (\Pi_{i=1}^k G_i) \tilde{x}_{k_0} + \sum_{i=1}^j (\Pi_{i-j+1}^k G_i) \Delta_{i|j}, \\
\tilde{y}_{k_{j+1}} &= (\Pi_{i=1}^k \hat{G}_i) \tilde{y}_{k_0} + \sum_{i=1}^j (\Pi_{i-j+1}^k \hat{G}_i) \hat{\Delta}_{i|j},
\end{align*}
\]

where \( G_k = (I - \Delta_k)A \) and \( \hat{G}_k = (I - \hat{\Delta}_k)\hat{A}_k, k \geq 1 \). Now under the hypothesis of Theorem 2.5

\[
G_k \to G = (I - \Delta_0)A = W(S_0 + W)^{-1}A, \quad k \to \infty,
\]

where \( \rho(G) = \rho(W(S_0 + W)^{-1}A) = \rho(AW(S_0 + W)^{-1}) < 1 \) by Lemma 4.3. Therefore, there is an \( r \geq 1 \) for which \( \|G^r\| < 1 \), and since \( G_k \to G \) and \( \hat{G}_k \to G \) with probability one by Corollaries 2.1 and 2.3 respectively, there exist \( \rho_0 < 1 \) and a (random) integer \( k_0 \geq 1 \) such that with probability one

\[
\max \{\|\Pi_{i=1}^k G_i\|, \|\Pi_{i=1}^k \hat{G}_i\|\} \leq \rho_0
\]

whenever \( j \geq k_0 \). In particular,

\[
\max \{\|\Pi_{i=1}^k G_i\|, \|\Pi_{i=1}^k \hat{G}_i\|\} \to 0 \quad \text{with probability one}
\]
as \( k \to \infty \). It follows that for any \( j_0 \geq 1 \)

\[
\begin{align*}
\lim_{i \to \infty} \sup n^{-1} \sum_{s=1}^n |\tilde{x}_{k_{j|s}}| \\
&\leq \lim_{i \to \infty} \sup n^{-1} \sum_{s=1}^n \sum_{s=1}^n \|\Pi_{i-j+1}^k \hat{G}_i \Delta_{j|s}\| \|\Pi_{i-j+1}^k G_i \Delta_{j|s}\| \|y_{j|s}\| \\
(5.1) &\leq \sup_{j \geq j_0} (\sum_{i=j_0}^j \|\Pi_{i-j+1}^k \hat{G}_i \Delta_{j|s}\| \|\Pi_{i-j+1}^k G_i \Delta_{j|s}\|) (\lim_{i \to \infty} \sum_{i=j_0}^j \|y_{j|s}\|) \\
&\leq \sup_{j \geq j_0} (\sum_{i=j_0}^j \|\Pi_{i-j+1}^k \hat{G}_i \Delta_{j|s}\| \|\Pi_{i-j+1}^k G_i \Delta_{j|s}\|) (\lim_{i \to \infty} \sum \|y_{j|s}\|) \\
&\to 0
\end{align*}
\]

where the final inequality follows as in (3.8). Moreover, since \( \hat{\Delta}_k \to \Delta_0 \leftarrow \Delta_k \) and \( \hat{G}_k \to G \leftarrow G_k \) as \( k \to \infty \), we have, for any fixed \( j' \geq 1 \) that

\[
\lim_{j \to \infty} \|\Pi_{i-j+1}^k \hat{G}_i \Delta_{j|s} - \Pi_{i-j+1}^k G_i \Delta_{j|s}\| = 0.
\]

It follows immediately that for any \( s \geq 1 \)

\[
\begin{align*}
\lim_{j \to \infty} (\sum_{s=j}^{j_0} \|\Pi_{i-j+1}^k \hat{G}_i \Delta_{j|s} - \Pi_{i-j+1}^k G_i \Delta_{j|s}\|) \\
(5.2) &\leq \lim \sup_{j \to \infty} \sum_{s=j}^{j_0} \|\Pi_{i-j+1}^k \hat{G}_i \Delta_{j|s} - \Pi_{i-j+1}^k G_i \Delta_{j|s}\| \\
&\leq 2\|\Delta_0\|\rho_0^s/(1 - \rho_0), \quad \text{with probability one}
\end{align*}
\]
which may be made arbitrarily small by proper choice of $s$. Theorem 2.5 follows easily from (5.1) and (5.2).

6. Numerical results. A computer program embodying the estimators described above gave the results in Table 1. In this program the linear system (1.1) and (1.2) was scalar with normal noise and parameters $\lambda = 0.9$, $V = 4.0$, and $W = 1.0$. The initial condition on $x_0$ was $x_0 = 100.0$. The program simulated the system (1.1) and (1.2) and computed the estimators $\hat{A}_n$, $\hat{V}_n$, $\hat{W}_n$, and $\hat{\Delta}$ over periods of time of length 20, 40, 60, 80, 100, and 200. Fifty runs were made for each of these time periods.

<table>
<thead>
<tr>
<th>Time Period $n$</th>
<th>$\hat{A}_n$ mean</th>
<th>$\hat{A}_n$ variance</th>
<th>$\hat{V}_n$ mean</th>
<th>$\hat{V}_n$ variance</th>
<th>$\hat{W}_n$ mean</th>
<th>$\hat{W}_n$ variance</th>
<th>$\hat{\Delta}$ mean</th>
<th>$\hat{\Delta}$ variance</th>
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<td>.899</td>
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<td>.274 $\times 10^{1}$</td>
<td>.481 $\times 10^{1}$</td>
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<td></td>
<td></td>
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</tr>
<tr>
<td>40</td>
<td>.899</td>
<td>$.110 \times 10^{-3}$</td>
<td>.331 $\times 10^{1}$</td>
<td>.364 $\times 10^{1}$</td>
<td></td>
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<tr>
<td>60</td>
<td>.900</td>
<td>$.141 \times 10^{-3}$</td>
<td>.348 $\times 10^{1}$</td>
<td>.170 $\times 10^{1}$</td>
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<td></td>
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<tr>
<td>80</td>
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<td>$.547 \times 10^{-4}$</td>
<td>.344 $\times 10^{1}$</td>
<td>.126 $\times 10^{1}$</td>
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<tr>
<td>100</td>
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<td>.362 $\times 10^{1}$</td>
<td>.170 $\times 10^{1}$</td>
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<td></td>
<td></td>
</tr>
<tr>
<td>200</td>
<td>.901</td>
<td>$.768 \times 10^{-4}$</td>
<td>.378 $\times 10^{1}$</td>
<td>.624</td>
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<table>
<thead>
<tr>
<th>Time Period $n$</th>
<th>$\hat{W}_n$ mean</th>
<th>$\hat{W}_n$ variance</th>
<th>$\hat{\Delta}_n$ mean</th>
<th>$\hat{\Delta}_n$ variance</th>
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</thead>
<tbody>
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<td>$.174 \times 10^{1}$</td>
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<tr>
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<td>$.110 \times 10^{1}$</td>
<td>$.232</td>
<td>.802</td>
<td>$.669 \times 10^{-1}$</td>
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</table>

Computation of $\Delta_n$ showed that it was stationary at the end of 20 time periods at $\Delta_n = 0.824$.

It was found that the parameter estimators are sensitive to the initial condition of the linear system. Occasionally when the system is at $x_0 = 0$ the fluctuations in the initial values of $\hat{A}_n$ cause the $B_n$, $B_n$, to assume extremely high values so that the corresponding means and variances of $\hat{V}_n$, and $\hat{W}_n$ display large dispersion. This problem does not arise when the initial conditions of the process differ from zero enough to give initial stability to $\hat{A}_n$.

REFERENCES


