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ON THE EXPANSION FOR EXPECTED SAMPLE SIZE IN NON-LINEAR RENEWAL THEORY

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An asymptotic expansion for the expected sample size in the non-linear renewal theorem is shown to hold under an alternative set of regularity conditions. The alternative set requires fewer moments than existing conditions in an important special case, but is more restrictive in other ways.

Let $X_1, X_2, \cdots$ be i.i.d. random variables with finite, positive mean and variance $0 < \mu$, $\sigma^2 < \infty$, and let S_n, $n \geq 0$, denote the random walk $S_0 = 0$ and $S_n = X_1 + \cdots + X_n$, $n \geq 1$. Further, let $\xi_1, \xi_2, \cdots$ be random variables for which ξ_n is independent of the sequence X_k, $k > n$, for every $n \geq 1$, and consider the process

$$Z_n = S_n + \xi_n, \qquad n \geq 1.$$

Let

$$\tau_a = \inf\{n \geq 1 : S_n > a\},$$
$$t_a = \inf\{n \geq 1 : Z_n > a\},$$

and

$$R_a = Z_{t_a} - a, \quad \text{on} \quad \{t_a < \infty\}, \qquad a \geq 0,$$

where the infimum of the empty set is ∞, by convention. Observe that $\tau = \tau_0$ is the first strict ascending ladder epoch, in the terminology of Feller (1966). It is well known that if X_1 does not have an arithmetic distribution, then $S_{\tau_a} - a$ has an asymptotic distribution H as $a \to \infty$, where

$$H\{dr\} = (1/E(S_\tau))\, P\{S_\tau > r\}\, dr, \qquad r > 0. \tag{1}$$

See, for example, Feller (1966, pages 354–355). Similarly, if X_1 has an arithmetic distribution with span d, then $S_{\tau_a} - a$ has a limiting distribution as a through multiple of d. In this case the limiting distribution H is discrete and assigns mass $H\{kd\} = (d/E(S_\tau))\, P\{S_\tau \geq kd\}$ to the points kd, $k \geq 1$.

Recently, Lai and Siegmund (1977), Hagwood (1980), and Lalley (1980) have given conditions under which R_a has a limiting distribution. The main result of Lai and Siegmund is stated for reference.

THEOREM 1. *Suppose that there is an α, $\frac{1}{2} < \alpha \leq 1$ for which*

$$a^{-\alpha}\left(t_a - \frac{a}{\mu}\right) \to_p 0, \quad as \quad a \to \infty, \tag{2}$$

and the following condition holds: for every $\varepsilon > 0$ there is a $\delta > 0$ for which

$$P\{\max_{0 \leq k \leq \delta n^\alpha} |\xi_{n+k} - \xi_n| \geq \varepsilon\} < \varepsilon \tag{3}$$

for all sufficiently large n. If X does not have an arithmetic distribution, then R_a has the limiting distribution (1) *as $a \to \infty$.*

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Hagwood (1980) and Lalley (1980) obtain related results in the arithmetic case under additional conditions on ξ_n, $n \geq 1$. The limiting distribution in the arithmetic case is more complicated than H, and will not be described.

The Condition (3) is most intuitive in the special case that $\alpha = 1$; for then it requires that ξ_n, $n \geq 1$, be uniformly continuous in probability, as in the original formulation of Anscombe's (1952) theorem. Observe that, if $\alpha = 1$, then Condition (2) will hold if

$$\max\{|\xi_1|, \cdots, |\xi_n|\} = o_p(n), \quad \text{as} \quad n \to \infty. \tag{4}$$

We will say that ξ_n, $n \geq 1$, are slowly changing iff they satisfy (3) with $\alpha = 1$ and (4).

In a later paper, Lai and Siegmund (1979) gave conditions under which powers of R_a are uniformly integrable and the expansion (12) for $E(t_a)$ is valid. In this note we give an alternative set of conditions under which the expansion for $E(t_a)$ is valid. We compare these conditions with those of Lai and Siegmund, after proving the result.

Let $\mathscr{A}_n = \sigma\{(X_k, \xi_k)\ k \leq n\}$ be the sigma-algebra generated by (X_k, ξ_k), $k \leq n$ for $n \geq 1$, and observe that $\mathscr{A}_n$ is independent of the sequence X_k, $k > n$, in view of the standing assumptions on ξ_k, $k \geq 1$. In the theorem below, we suppose that there are $\mathscr{A}_n$-measurable events A_n, $n \geq 1$, constants $f(n)$, $n \geq 1$, and $\mathscr{A}_n$-measurable random variables V_n, $n \geq 1$, for which:

$$\textstyle\sum_{n=1}^{\infty} P(\cup_{k=n}^{\infty} A_k') < \infty; \tag{5}$$

$$\xi_n = f(n) + V_n \quad \text{on} \quad A_n, \qquad n \geq 1; \tag{6}$$

$$\sup_{n\geq 1} \max_{0\leq k\leq n\delta} |f(n+k) - f(n)| \to 0, \quad \text{as} \quad \delta \to 0; \tag{7}$$

$$\max_{0\leq k\leq n} |V_{n+k}|,\ n \geq 1, \quad \text{are uniformly integrable;} \tag{8}$$

$$\textstyle\sum_{n=1}^{\infty} P\{V_n \leq -n\varepsilon\} < \infty, \qquad \exists \varepsilon, \qquad 0 < \varepsilon < \mu; \tag{9}$$

$$V_n \text{ converges in distribution to a random variable } V, \text{ as } n \to \infty; \tag{10}$$

and

$$P\{t_a < \varepsilon a\} = o(1/a), \qquad \exists \varepsilon > 0, \quad \text{as} \quad a \to \infty. \tag{11}$$

In Theorem 2 below, the constants $f(n)$, $n \geq 1$, are extended to a function on $[1, \infty)$ by linear interpolation.

THEOREM 2. *Suppose that Conditions* (5) *through* (11) *are satisfied and that* V_n, $n \geq 1$, *are slowly changing. Then there are events* $B = B_a$ *for which* $P(B) \to 1$, $R_a I_B$, $a > 0$, *are uniformly integrable and*

$$E(t_a) = \frac{1}{\mu}\left\{a + \int_B R_a\, dP - \left[f\left(\frac{a}{\mu}\right) + E(V)\right]\right\} + o(1), \quad \textit{as} \quad a \to \infty. \tag{12}$$

COROLLARY 1. *If* X_1 *is non-arithmetic, then* $\int_B R_a\, dP$ *may be replaced by*

$$\rho = E(S_\tau^2)/2E(S_\tau),$$

the mean of the distribution H, *in* (12).

PROOFS. If V_n, $n \geq 1$, are slowly changing, then so are ξ_n, $n \geq 1$, by (5), (6), and (7). So, the corollary follows directly from Theorem 1 and the uniform integrability asserted in Theorem 2.

In the proof of Theorem 2, we write $t = t_a$. By (11) and Lemma 3 below there are ε_1 and ε_2 for which $0 < \varepsilon_1 < \varepsilon_2 < \infty$ and

$$aP\{t < n_1\} + \int_{t>n_2} t\,dP \to 0, \qquad \text{as} \quad a \to \infty, \tag{13}$$

where $n_1 = n_1(a) = [\varepsilon_1 a]$, $n_2 = n_2(a) = [\varepsilon_2 a]$, and $[-]$ denotes the greatest integer which is

less than or equal to $-$. Next, let $N = N(a) = [\mu^{-1}a]$; let $b = b(a) = a - f(N)$; and let

$$B = B_a = \{n_1 \le t \le n_2, \tau_b > n_1\} \cap \cap_{k=n_1}^{\infty} A_k, \qquad a > 0.$$

Then it follows easily from (5), (13), and the standing assumption that X_1 has a finite variance that $P(B') = o(1/a)$ as $a \to \infty$. So,

$$\int_B S_t \, dP = \int_B (a + R_a - \xi_t) \, dP = a + \int_B (R_a - \xi_t) \, dP + o(1), \qquad \text{as} \quad a \to \infty.$$

We need

$$\int_B S_t \, dP = \mu E(t) + o(1)$$

$$\int_B \xi_t \, dP = f\left(\frac{a}{\mu}\right) + E(V) + o(1), \tag{14}$$

and the uniform integrability of $R_a I_B$, $a > 0$. The first of these assertions follows from Wald's Lemma and (13) by an argument which is nearly identical to the proof of Lemma 4 in Lai and Siegmund (1979). The others are estblished in Lemmas 1 and 2 below.

LEMMA 1. *Relation* (14) *holds.*

PROOF. Clearly, $\xi_t = \xi_N + (\xi_t - \xi_N)$ and

$$\int_B \xi_N \, dP = f\left(\frac{a}{\mu}\right) + E(V) + o(1), \qquad \text{as} \quad a \to \infty.$$

Since ξ_n, $n \ge 1$, are slowly changing, $t/N \to 1$ and $(\xi_t - \xi_N) \to 0$ in probability. So, it suffices to show that $(\xi_t - \xi_N)\mathrm{I}_B$ are uniformly integrable; and this follows easily from (7), (8), and

$$|\xi_t - \xi_N| \mathrm{I}_B \le \max_{n_1 \le k \le n_2} |f(k) - f(N)| + 2\max_{n_1 \le k \le n_2} |V_k|.$$

Observe that the proof of Lemma 1 shows that the random variables $[\xi_t - f(N)]I_B = V_N I_B + (\xi_t - \xi_N) I_B$, $a > 0$, are uniformly integrable too. In fact,

$$M_a = \max_{n_1 \le k \le n_2} |\xi_k - f(N)|, \qquad a > 0$$

are uniformly integrable.

LEMMA 2. $R_a I_B$, $a > 0$, *are uniformly integrable.*

PROOF. Since $[\xi_t - f(N)]I_B$, $a > 0$, are uniformly integrable, it suffices to show the uniform integrability of the positive parts of

$$R_a^* = [R_a - (\xi_t - f(N))]I_B = (S_t - b)I_B.$$

Let

$$C = C_a(x) = \{M_a \le x\}, \qquad a, x > 0.$$

Then

$$P\{R_a^* > 2x, BC\} = \sum_{n=n_1}^{n_2} P\{t = n, S_n > b + 2x, BC\}$$

$$\le \sum_{n=n_1}^{n_2} P\{t \ge n, S_n > b + 2x, BC\}, \qquad a, x > 0.$$

Now if B and C both occur and if $t \ge n$, then $\tau_{b+x} \ge n$ for all $x > 0$; for B requires $\tau_{b+x} \ge \tau_b \ge n_1$, and $C\{t \ge n\}$ requires $S_k = Z_k - \xi_k = Z_k - f(N) - [\xi_k - f(N)] \ge a - f(N) + x$

$= b + x$ for $n_1 \le k \le n_2$. So, for $n_1 \le n \le n_2$,

$$P\{t \ge n, S_n > b + 2x, BC\} \le P\{\tau_{b+x} \ge n, S_n > (b + x) + x, BC\}$$
$$\le P\{\tau_{b+x} = n, S_n - (b + x) > x\};$$

and

$$P\{R_a^* > 2x, BC\} \le \textstyle\sum_{n=n_1}^{n_2} P\{\tau_{b+x} = n, S_n - (b + x) > x\}$$
$$\le P\{S_{\tau_{b+x}} - (b + x) > x\} \le \sup_{c>0} P\{S_{\tau_c} - c > x\}$$

for $x > 0$. The latter function is integrable w.r.t x over $(0, \infty)$, by Theorem 4 of Lorden (1970) or a careful reading of the proof of Corollary 1 in Lai and Siegmund (1979). The uniform integrability of R_a^* now follows easily from that of M_a, $a > 0$, and the relation $P\{R_a^* > 2x, B\} \le P\{R_a^* > 2x, BC\} + P(M_a > x\}$, $a, x > 0$.

LEMMA 3. *There are ε_1 and ε_2 for which $0 < \varepsilon_1 < \varepsilon_2 < \infty$ and* (13) *holds.*

PROOF. The existence of ε_1 for which $aP\{t < \varepsilon_1 a\} \to 0$ as $a \to \infty$ follows directly from (11). To establish the existence of ε_2, let ε be as in (9), $0 < \varepsilon < \mu$; let $\delta > 0$ be so small that $\varepsilon + \delta < \mu$; and let $\varepsilon_2 = 2/(\mu - \varepsilon - \delta)$. Then, for $n > K_a = [\frac{1}{2}\varepsilon_2 a]$, $a - n\mu < -n(\varepsilon + \delta)$, so that

$$P\{t > n\} \le P\{S_n - n\mu + \xi_n < a - n\mu\} \le P\{S_n - n\mu < -n\delta\} + P\{\xi_n < -n\varepsilon\}. \tag{15}$$

Denote the right side of (15) by γ_n, $n \ge 1$. Then γ_n, $n \ge 1$, are summable by (9) and the standing assumption that X_1 has a finite variance. See, for example, Baum and Katz (1965). It follows that

$$\int_{t>2K_a} t\,dP \le 2\int_{t>2K_a} (t - K_a)\,dP \le 2\textstyle\sum_{n=K_a}^{\infty} \gamma_n = o(1), \qquad \text{as} \quad a \to \infty.$$

This completes the proof of Theorem 2.

The Condition (7) of Theorem 2 is clearly stronger than the corresponding Condition (13) of Lai and Siegmund. On the other hand, Theorem 2 imposes fewer moment conditions than Theorem 3 of Lai and Siegmund (1979)—at least, in the context of the following important special case.

EXAMPLE. *First Exit From a Square Root Boundary.* Let $Y_1, Y_2, \cdots$ be i.i.d. random variables with finite mean $\nu \ne 0$ and finite positive variance; and let

$$t_a = \inf\{n \ge 1 : |Y_1 + \cdots + Y_n| > \sqrt{(2an)}\}, \qquad a > 0.$$

Then t_a has the right form with $Z_n = \frac{1}{2}n\bar{Y}_n^2$ and $\bar{Y}_n = (Y_1 + \cdots + Y_n)/n$, $n \ge 1$. Now, Z_n may be written in the form $Z_n = S_n + \xi_n$, where

$$S_n = \tfrac{1}{2}n\nu^2 + n\nu(\bar{Y}_n - \nu) \quad \text{and} \quad \xi_n = \tfrac{1}{2}n(\bar{Y}_n - \nu)^2, \qquad n \ge 1.$$

It is easy to check that the standing assumptions and Conditions (3) through (10) are satisfied, under the sole assumption that $E(Y_1^2) < \infty$. Indeed, letting A_n be the entire sample space and $f(n) = 0$ for $n \ge 1$, Conditions (5), (6), (7), and (9) are clearly satisfied. Condition (4) follows from the S.L.L.N.; the asymptotic distribution of $V_n = \xi_n$ is a multiple of Chi squared in (10); and Conditions (3) and (8) follows from standard maximal inequalities.

Condition (11) is satisfied if $E|Y_1|^\alpha < \infty$ for some $\alpha > 2$. To see this first observe that $\sqrt{(2an)} - n\nu > \sqrt{an}$ for all $n < \varepsilon a$ for sufficiently small $\varepsilon > 0$. Thus, for such ε, $t < \varepsilon a$ implies that

$$B_k = \{\max_{n \le 2^k} |Y_1 + \cdots + Y_n - n\nu| > \sqrt{(a2^{k-1})}\}$$

occurs for some $k \le K_a = [\log_2(\varepsilon a) + 1]$. By the submartingale inequality,

$$P(B_k) \le \left(\frac{2}{a}\right)^{\alpha/2} E\,|\,2^{-k/2}(Y_1 + \cdots + Y_{2^k} - 2^k\nu)\,|^\alpha, \qquad k \ge 1; \tag{16}$$

and, by Theorem 3 of von Bahr (1965), the expectation in (16) remains bounded as $k \to \infty$. That condition (11) is satisfied follows immediately.

Thus, if $E\,|\,Y_1\,|^\alpha < \infty$ for some $\alpha > 2$, then the expansion (12) holds for t_a; and if $X_1 = \frac{1}{2}\nu^2 + \nu(Y_1 - \nu)$ has a non-arithmetic distribution, then ρ may be substituted for $\int_B R_a\,dP$. This example has been considered in detail by Woodroofe (1976) and Lai and Siegmund (1979). Theorem 3 of Lai and Siegmund appears to require that Y_1 have a fourth moment.

REFERENCES

[1] ANSCOMBE, F. J. (1952). Large sample theory of sequential estimation. *Proc. Cambridge Philos. Soc.* **48** 600–607.

[2] BAUM, L. and KATZ, M. (1965). Convergence rates in the low of large numbers. *Trans. Amer. Math. Soc.* **120** 108–123.

[3] FELLER, WILLIAM (1966). *An Introduction to Probability Theory and its Applications, Vol. 2.* Wiley, New York.

[4] HAGWOOD, C. (1980). A non-linear renewal theorem for discrete random variables. *Comm. Statist.* **A9** No. 16, 1677–1698.

[5] LAI, T. L. and SIEGMUND, D. (1977). A non-linear renewal theory with applications to sequential analysis, I. *Ann. Statist.* **5** 946–954.

[6] LAI, T. L. and SIEGMUND, D. (1979). Non-linear enewal theory with applications to sequential analysis II. *Ann. Statist.* **7** 60–76.

[7] LALLEY, S. (1980). *Repeated likelihood ratio tests for curved exponential families.* Ph.D. Thesis, Stanford University.

[8] LORDEN, G. (1970). On the excess over the boundary. *Ann. Math. Statist.* **41** 520–526.

[9] WOODROOFE, M. (1976). A renewal theorem for curved boundaries and moments of first passage times. *Ann. Probability* **4** 67–80.

[10] VON BAHR, B. (1965). On the convergence of moments in the central limit theorem. *Ann. Math. Statist.* **36** 808–818.

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