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Non-linear Renewal Theory with Stationary Perturbations

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ABSTRACT

A non-linear renewal theorem is obtained from random walks that are perturbed by an approximately stationary sequence. As corollaries, the limiting joint distribution of the excess over the boundary and last perturbation are obtained along with an approximation to expected first passage times. The results are illustrated by an analysis of a sequential probability ratio test when data are subject to both censoring and staggered entry.

Key Words: Asymptotic distributions; Censoring; Likelihood function; Random walks; Uniform integrability.

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INTRODUCTION

Sequential problems that involve both censoring and staggered entry lead to processes that are random walks perturbed, not by slowly changing terms, but (approximately) stationary ones. To see how, consider the following simple example. Suppose that patients arrive for treatment at times $0 = \tau_0 < \tau_1 < \tau_2 < \dots$, are treated, and live for exponentially distributed periods $Y_1, Y_2, \dots$ thereafter, with an unknown failure rate θ. Then the data available at time τ_n consist of

$$Y_{nk} = \min[Y_k, \tau_n - \tau_{k-1}],$$

$$\Delta_{nk} = 1_{\{Y_k \le \tau_n - \tau_{k-1}\}}$$

for $k = 1, \dots, n$. The log-likelihood function given $Y_{n1}, \dots, Y_{nn}$, $\Delta_{n1}, \dots, \Delta_{nn}$ is

$$\ell_n(\theta) = K_n \log(\theta) - \theta T_n,$$

where $K_n = \Delta_{n1} + \cdots + \Delta_{nn}$ and $T_n = Y_{n1} + \cdots + Y_{nn}$ are the number of failures and the total time on test at time τ_n. The sequential probability ratio test for testing $H_0: \theta = 1$ vs. $H_1: \theta = \delta \neq 1$ may be described as follows: Let

$$\Lambda_n = \ell_n(\delta) - \ell_n(1),$$

$$N_a = \inf\{n \ge 1 : \Lambda_n > a\},$$

and

$$M_b = \inf\{n \ge 1 : \Lambda_n < -b\}$$

for $a, b \ge 0$. The test takes a sample of size $\min(N_a, M_b)$ and rejects H_0 in favor of H_1 iff $N_a < M_b$. The dominant term in the type I error probability is then

$$P_1[N_a < \infty] = \int_{\{N_a < \infty\}} e^{-\Lambda_{N_a}} dP_\delta = e^{-a} E_\delta[e^{-(\Lambda_{N_a} - a)}],$$

and the asymptotic distribution of $\Lambda_{N_a} - a$ is of interest. Following the procedure in Lai and Siegmund,[3,4] it seems natural to write Λ_n as a perturbed random walk. This is not difficult, since

$$\Lambda_n = n \log(\delta) - (\delta - 1)(Y_1 + \cdots + Y_n) + \tilde{\xi}_n,$$

where

$$\tilde{\xi}_n = (\delta - 1)\sum_{j=1}^{n}[Y_{n-j+1} - (\tau_n - \tau_{n-j})]^+ - \log(\delta)\sum_{j=1}^{n} 1_{\{Y_{n-j+1} > \tau_n - \tau_{n-j}\}},$$

1_B denotes the indicator function of B, and $x^+ = \max(x,\ 0)$. Unfortunately, $\tilde{\xi}_n$ is not slowly changing, as required by Lai and Siegmund. However, if the arrival times τ_k form a renewal process (so that $\tau_k - \tau_{k-1}$, $k = 0, \pm 1, \pm 2, \ldots$ are i.i.d.), then $\tilde{\xi}_n$ does have some structure, since

$$\tilde{\xi}_n \approx (\delta - 1)\sum_{j=1}^{\infty}[Y_{n-j+1} - (\tau_n - \tau_{n-j})]^+ - \log(\delta)\sum_{j=1}^{\infty} 1_{\{Y_{n-j+1} > \tau_n - \tau_{n-j}\}} = \xi_n,$$

say, and the right side is a stationary process.

The proceeding is intended to motivate the study of renewal theory for processes of the form

$$Z_n = S_n + \xi_n, \tag{1}$$

where S_n is a random walk with a positive drift μ, say, and ξ_n is a stationary sequence with common marginal distribution function G, say. The goal of the article is to contribute to such a study. A main result is that

$$\lim_{a\to\infty}\sum_{n=1}^{\infty} P\{\xi_n \le c, a < Z_n \le a + b\} = \frac{b}{\mu}G(c) \tag{2}$$

for $0 < b < \infty$ and continuity points c of G, under modest conditions. Denote the first passage times and excesses by

$$t_a = \inf\{n \ge 1 : Z_n > a\} \tag{3}$$

$$R_a = Z_{t_a} - a. \tag{4}$$

Then the asymptotic joint distribution of ξ_{t_a} and R_a may be obtained as from Eq. (2). It is shown that ξ_{t_a} and R_a have asymptotic joint distribution function,

$$H(c,r) = \frac{1}{\mu}\int_0^r P[\xi_0 \le c, M \ge u]\, du \tag{5}$$

where

$$M = \inf_{k<0} Z_k \tag{6}$$

and

$$Z_k = X_{k+1} + \cdots + X_0 + (\xi_0 - \xi_k) \tag{7}$$

for $k \leq -1$. Under additional moment conditions, it is shown that R_a and ξ_{t_a} are uniformly integrable in $a > 0$ and that the main results continue to hold for approximately stationary sequences. The main results are specialized to the exponential case in Sec. 2, and the accuracy of the resulting approximations is assessed using simulation. Formal statements and proofs of the main results are presented in Secs. 4 and 5. Sec. 3 contains some preliminary lemmas.

The asymptotic (marginal) distribution of R_a is not new. It may be obtained from Lalley's[5] general renewal theorem for sums of stationary processes. Relations (2), (5), and the uniform integrability are new, however (to the best of the authors' knowledge), and the derivation of the asymptotic distribution differs greatly from Lalley's. The work of Melfi[6,7] and Su[8,9] is related.

THE EXPONENTIAL CASE

The limiting distribution (5) of R_a and ξ_{t_a} suggests approximations to the actual distributions. The use of these approximations is illustrated in this section, and their accuracy assessed, in the exponential model with $\tau_k = k$. Then

$$X_k = \log(\delta) - (\delta - 1)Y_k,$$

$$\xi_k = \sum_{j=1}^{\infty} [(\delta - 1)[Y_{k-j+1} - j]^+ - \log(\delta)1_{\{Y_{k-j+1} > j\}}],$$

where Y_k are i.i.d. exponentially distributed random variables and $\delta \neq 1$, and $S_n = X_1 + \cdots + X_n$. As indicated in Introduction, there is special interest in the distribution of R_a when $\theta = \delta$. Then the mean of X_k is

$$\mu = \frac{1}{\delta} - 1 - \log\left(\frac{1}{\delta}\right) > 0.$$

The distribution function H of Eq. (5) does not simplify, even in the exponential case. It can be approximated by simulation, however. In these simulations, the infimum M in Eq. (6) was truncated after 5000. Define $\tilde{t}_a$ and $\tilde{R}_a$ by Eqs. (1), (3), and (4) with ξ_n replaced by $\tilde{\xi}_n$. Then the means of the asymptotic distribution of $\tilde{\xi}_{\tilde{t}_a}$ and $\tilde{R}_a$ are

$$\nu = \int_R \int_0^\infty cH\{dcdr\} = \frac{1}{\mu} E(M^+ \xi_0)$$

and

$$\rho = \int_R \int_0^\infty rH\{dcdr\} = \frac{1}{2\mu} E\,[(M^+)^2];$$

and

$$\lim_{a\to\infty} E(e^{-\tilde{R}_a}) = E\left[\frac{1 - e^{-M^+}}{\mu}\right].$$

The final rows of Tables 1–3 below list Monte Carlo estimates of these values, based on 50,000 replications for $\delta = 0.5$, 0.75, 1.25, and 1.5.

To assess the speed of convergence, $E_\delta(\tilde{\xi}_{\tilde{t}_a})$, $E_\delta(\tilde{R}_a)$, and $E_\delta(e^{-\tilde{R}_a})$ were simulated for selected values of a. The differences are at most two standard deviations for $a \geq 4$ in all but one case, and in many cases for $a \geq 2$. The densities of R_a are compared to their limiting value in Fig. 1.

Table 1. Convergence rate of average excess.

	$\delta = 0.5$	$\delta = 0.75$	$\delta = 1.25$	$\delta = 1.5$
a	$\hat{E}(\tilde{R}_a) \pm \tilde{\sigma}(\tilde{R}_a)$	$\hat{E}(\tilde{R}_a) \pm \hat{\sigma}(\tilde{R}_a)$	$\hat{E}(\tilde{R}_a) \pm \hat{\sigma}(\tilde{R}_a)$	$\hat{E}(\tilde{R}_a) \pm \hat{\sigma}(\tilde{R}_a)$
0.5	0.5649 ± 0.0017	0.2153 ± 0.0007	0.0753 ± 0.0002	0.1403 ± 0.0004
1	0.4870 ± 0.0014	0.2162 ± 0.0008	0.0756 ± 0.0002	0.1392 ± 0.0004
2	0.5377 ± 0.0019	0.2199 ± 0.0008	0.0758 ± 0.0002	0.1388 ± 0.0004
4	0.5598 ± 0.0020	0.2227 ± 0.0008	0.0757 ± 0.0002	0.1394 ± 0.0004
8	0.5724 ± 0.0021	0.2216 ± 0.0008	0.0758 ± 0.0002	0.1393 ± 0.0004
∞	0.5751 ± 0.0060	0.2248 ± 0.0040	0.0758 ± 0.0009	0.1391 ± 0.0013

Table 2. Convergence rate of $E(e^{-\tilde{R}a})$.

	$\delta = 0.5$	$\delta = 0.75$	$\delta = 1.25$	$\delta = 1.5$
a	$\hat{E}(e^{-\tilde{R}_a}) \pm \hat{\sigma}_{e^{-\tilde{R}_a}}$	$\hat{E}(e^{-\tilde{R}_a}) \pm \hat{\sigma}_{e^{-\tilde{R}_a}}$	$\hat{E}(e^{-\tilde{R}_a}) \pm \hat{\sigma}_{e^{-\tilde{R}_a}}$	$\hat{E}(e^{-\tilde{R}_a}) \pm \hat{\sigma}_{e^{-\tilde{R}_a}}$
0.5	0.6094 ± 0.0010	0.8155 ± 0.0005	0.9288 ± 0.0002	0.8730 ± 0.0004
1	0.6433 ± 0.0008	0.8171 ± 0.0006	0.9285 ± 0.0002	0.8741 ± 0.0004
2	0.6319 ± 0.0010	0.8145 ± 0.0006	0.9283 ± 0.0002	0.8744 ± 0.0004
4	0.6228 ± 0.0010	0.8123 ± 0.0006	0.9284 ± 0.0002	0.8739 ± 0.0004
8	0.6176 ± 0.0010	0.8133 ± 0.0006	0.9283 ± 0.0002	0.8740 ± 0.0004
∞	0.6155 ± 0.0040	0.8196 ± 0.0103	0.9305 ± 0.0097	0.8717 ± 0.0065

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Table 3. Convergence rate of $E(\tilde{\xi}_{\tilde{t}_a})$.

	$\delta=0.5$	$\delta=0.75$	$\delta=1.25$	z$\delta=1.5$
a	$\hat{E}(\tilde{\xi}_{\tilde{t}_a})\pm\hat{\sigma}(\tilde{\xi}_{\tilde{t}_a})$	$\hat{E}(\tilde{\xi}_{\tilde{t}_a})\pm\hat{\sigma}(\tilde{\xi}_{\tilde{t}_a})$	$\hat{E}(\tilde{\xi}_{\tilde{t}_a})\pm\hat{\sigma}(\tilde{\xi}_{\tilde{t}_a})$	$\hat{E}(\tilde{\xi}_{\tilde{t}_a})\pm\hat{\sigma}(\tilde{\xi}_{\tilde{t}_a})$
0.5	-0.5843 ± 0.0062	-0.0950 ± 0.0021	0.0000 ± 0.0000	0.0000 ± 0.0000
1	-0.6724 ± 0.0066	-0.0993 ± 0.0022	0.0000 ± 0.0000	0.0000 ± 0.0000
2	-0.7409 ± 0.0070	-0.1021 ± 0.0022	0.0000 ± 0.0000	0.0000 ± 0.0000
4	-0.8315 ± 0.0074	-0.1004 ± 0.0022	0.0000 ± 0.0000	0.0000 ± 0.0000
8	-0.8688 ± 0.0076	-0.1031 ± 0.0022	0.0000 ± 0.0000	0.0000 ± 0.0000
∞	-0.8458 ± 0.0171	-0.1137 ± 0.0076	0.0000 ± 0.0000	0.0000 ± 0.0000

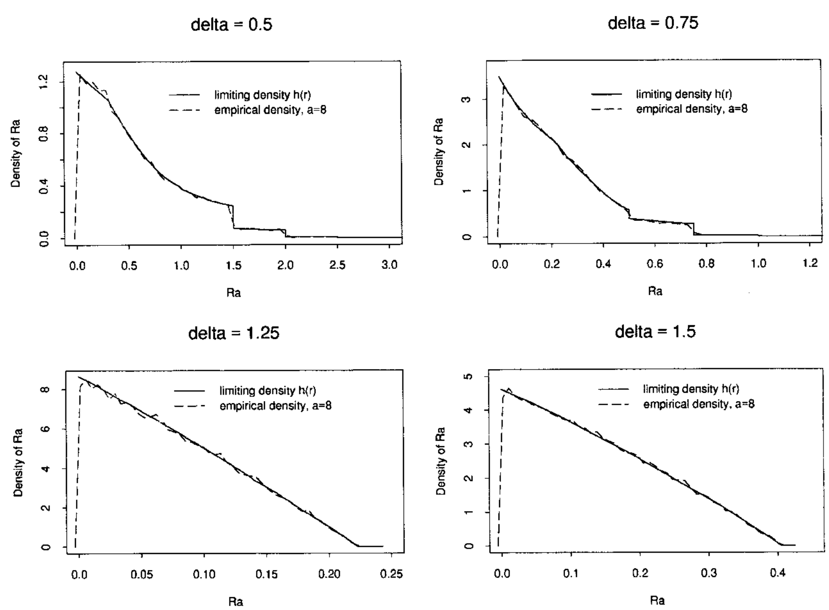

Figure 1. Limiting and empirical density of excess.

PRELIMINARIES

For the remainder of the article, $\ldots W_{-1}, W_0, W_1, W_2, \ldots$ denote i.i.d. random elements with values in a Polish space, $\mathcal{W}$ say, and $X_k=\varphi(W_k)$, where φ: $\mathcal{W}\to\mathbb{R}$ is a Borel measurable function. Let Q denote the (common) marginal distribution of the W_k; let F denote the marginal distribution of the X_k; and suppose throughout that F is a non-arithmetic with a finite, positive mean μ and variance σ^2. Next, let ξ_n be a sequence of the form

$$\xi_n = \xi(W_n, W_{n-1}, \ldots), \tag{8}$$

where ξ is a measurable function on $\mathcal{W}^{\mathbb{N}}$ and $\mathbb{N} = \{0,1,2,\ldots\}$; let G be the marginal distribution function of ξ_n; suppose throughout that G has a finite mean; and define Z_n by Eq. (1) with $S_n = X_1 + \cdots + X_n$. The exponential case is of this form with $W_k = Y_k$.

Several lemmas are needed. Of these, Lemmas 2 and 4 are both simple and crucial to subsequent developments. Given $0 < \eta < 1$, let

$$m = \left\lfloor \left(\frac{1-\eta}{\mu}\right) a \right\rfloor, \tag{9}$$

and

$$\Delta_0(\eta, a) = \sum_{n=0}^{m} P[Z_n > a], \tag{10}$$

where $\lfloor x \rfloor$ is the greatest integer that is less than or equal to x.

Lemma 1. *If* $0 < \eta < 1$, *then* $\lim_{a\to\infty} \Delta_0(\eta, a) = 0$.

Proof. The Baum–Katz[1] Inequalities are used in the following form: If $c > 0$ and n and r are positive integers, then

$$P\left[\max_{1\le k\le n} |S_k - k\mu| \ge c\right] \le n\bar{F}\left(\frac{c}{2r}\right) + 2\left(\frac{4r^2 n\sigma^2}{c^2}\right)^r,$$

where $\bar{F}(c) = F(-c) + 1 - F(c-)$. See Chow and Teicher[2] (pp. 373–375) for a derivation. So, for $n \le m = \lfloor (1-\eta)a/\mu \rfloor$,

$$\begin{aligned} P[Z_n > a] &\le P\left[S_n - n\mu > \frac{1}{2}\eta a\right] + P\left[\xi_n > \frac{1}{2}\eta a\right] \\ &\le n\bar{F}\left(\frac{\eta a}{8}\right) + 2\left(\frac{64 n\sigma^2}{\eta^2 a^2}\right)^2 + \bar{G}\left(\frac{1}{2}\eta a\right), \end{aligned}$$

and the lemma follows by summing this relation over $n \le m$. ◇

Let U denote the renewal measure for the random walk $S_0, S_1, S_2, \ldots$; that is,

$$U\{B\} = \sum_{n=0}^{\infty} P[S_n \in B]$$

for Borel sets $B \subseteq \mathbb{R}$. Then $\lim_{a\to\infty} U\{(a,a+b]\} = b/\mu$, by the Renewal Theorem. In addition, there is a constant $C_0 = C_0(F)$ for which $U\{(a,a+b]\} \le C_0(1+b)$ for all $a \in \mathbb{R}$ and $b > 0$.

Lemma 2. *If* $m \ge 1$ *is any integer*, $0 < a,b < \infty$, *and* $B \subseteq \mathcal{W}^m$ *is a Borel set*, *then*

$$\sum_{n=m}^{\infty} P[(W_n, \ldots, W_{n-m+1}) \in B,\ a < S_n \le a+b] = \int_B U\{(a-s,\ a-s+b]\} Q^m\{dw_1 \cdots dw_m\}, \tag{11}$$

where $s = \varphi(w_1) + \cdots + \varphi(w_m)$, *and* $Q^m = Q \times \cdots \times Q$ *denotes the product measure*. *Moreover*, *if* m *is as in Eq.* (9), *then*

$$\lim_{a\to\infty} \left| \sum_{n=0}^{\infty} P[(W_n, \ldots, W_{n-m+1}) \in B,\ a < S_n \le a+b] - \frac{b}{\mu} Q^m\{B\} \right| = 0$$

uniformly with respect to Borel sets $B \subseteq \mathcal{W}^m$ *for each fixed* $0 < b < \infty$.

Proof. By independence and symmetry

$$P[(W_n, \ldots, W_{n-m+1}) \in B,\ a < S_n \le a+b] = \int_B P[a-s < S_{n-m} \le a-s+b] Q^m\{dw_1 \cdots dw_m\}$$

for $n \ge m$. The first assertion follows by summing this relation over $n \ge m$. If m is an in Eq. (9) then

$$\left| \sum_{n=0}^{\infty} P[(W_n, \ldots, W_{n-m+1}) \in B, a < S_n \le a+b] - \frac{b}{\mu} Q^m\{B\} \right| \le \int_B \left| U\{(a-s,\ a-s+b]\} - \frac{b}{\mu} \right| Q^m\{dw_1 \cdots dw_m\} + \Delta_0(\eta, a)$$

Let $\varepsilon > 0$ be so small that $(1+\varepsilon)(1-\eta) < 1$ and let $a' = [1-(1+\varepsilon)(1-\eta)]a$. Then, dividing the last integral into the ranges $s \le (1+\varepsilon)m\mu$ and $s > (1+\varepsilon)m\mu$ shows that previous line is at most

$$\sup_{r \ge a'} \left| U\{(r, r+b]\} - \frac{b}{\mu} \right| + C_0(1+b)P[S_m > m(1+\varepsilon)\mu] + \Delta_0(\eta, a),$$

which is independent of B and approaches 0 as $a \to \infty$. The second assertion of the lemma follows directly. $\diamond$

For the next two lemmas, let $N\{B\} = \sum_{n=0}^{\infty} 1_B(S_n)$, so that $U\{B\} = E[N\{B\}]$. Also, denote elements of $\mathcal{W}^{\mathbb{N}}$ by $w = (w_0, w_{-1}, \ldots)$, and let $\mathbf{W}_n = (W_n, W_{n-1}, \ldots)$.

Lemma 3. *For any $b > 0$ and integer $k \geq 1$,* $\sup_{a\in\mathbb{R}} E\{N(a,a+b]^k\} < \infty$.

Proof. Expanding the product shows that $(N\,(a,\, a+b])^k \leq R_k$, where

$$R_k = k! \sum_{n_1=0}^{\infty} \sum_{n_2=n_1}^{\infty} \cdots \sum_{n_k=n_{k-1}}^{\infty} 1_{(a,a+b]}(S_{n_1}) \times \cdots \times 1_{(a,a+b]}(S_{n_k}).$$

Then conditioning on $X_1, \ldots, X_{n_{k-1}}$ and using $U(a - S_{n_{k-1}}, a - S_{n_{k-1}} + b] \leq C_0(1+b)$, shows that

$$\begin{aligned} E(R_k) &\leq k! E\left[\sum_{n_1=0}^{\infty} \cdots \sum_{n_{k-1}=n_{k-2}}^{\infty} 1_{(a,a+b]}(S_{n_1}) \times \cdots \times 1_{(a,a+b]}(S_{n_{k-1}}) C_0(1+b)\right] \\ &\leq k C_0(1+b) E(R_{k-1}) \end{aligned}$$

for $k \geq 2$, from which the lemma follows by induction. $\diamond$

Lemma 4. *For every $0 < b < \infty$ and $0 < \alpha < 1$, there are constants C_1 and C_2 for which*

$$\sum_{n=0}^{\infty} P[\mathbf{W}_n \in B,\, a < S_n \leq a+b] \leq C_1 P[\mathbf{W}_0 \in B]^{\alpha} \tag{12}$$

and

$$\sum_{n=0}^{\infty} P[\mathbf{W}_n \in B,\, a < Z_n \leq a+b] \leq C_2 \left[\int_{\mathbf{W}_0 \in B} (1 + |\xi_0|)^{\beta} dP\right]^{\alpha}, \tag{13}$$

for all Borel sets $B \subseteq \mathcal{W}^{\mathbb{N}}$ and $a \geq 0$, where $\beta = (1-\alpha^2)/\alpha$.

Proof. Let $q = 1/\alpha$ and $p = 1/(1-\alpha)$, so that p and q are conjugate values.

For Eq. (12), let $S_{-n} = X_{-n+1} + \cdots + X_0$ for $n \geq 0$. Then,

$$P[\mathbf{W}_n \in B,\, a < S_n \leq a+b] = P[\mathbf{W}_0 \in B,\, a < S_{-n} \leq a+b]$$

by stationarity. Summing this relation over n shows that the left side of Eq. (12) is

$$\sum_{n=0}^{\infty} P[\mathbf{W}_n \in B,\ a < S_n \leq a+b] = \int_{\{\mathbf{W}_0 \in B\}} \tilde{N}\{(a,a+b]\}\, dP, \tag{14}$$

where $\tilde{N}\{A\} = \sum_{n=0}^{\infty} 1_A(S_{-n})$. Clearly $\tilde{N}\{A\}$ has the same distribution as $N\{A\}$ for any Borel set $A \subseteq \mathbb{R}$ and, therefore, $C_1^p := \sup_{a\in\mathbb{R}} \times E\{\tilde{N}(a,\ a+b]^p\} < \infty$. Relation (12) now follows since the integral on the right side of Eq. (14) is at most

$$E\{\tilde{N}(a,\ a+b]^p\}^{1/p} P[\mathbf{W}_0 \in B]^{1/q} \leq C_1 P[\mathbf{W}_0 \in B]^{\alpha},$$

by Hölder's Inequality.

For Eq. (13), let $\beta = (1-\alpha^2)/\alpha$ and let C_1' be as in Eq. (12) with b replaced by $b+1$. Then left side of Eq. (13) at most

$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{k=-\infty}^{\infty} P[\mathbf{W}_n \in B,\ k-1 < \xi_n \leq k,\ a-k < S_n \leq a+b-k+1] \\
&\quad \leq 2C_1' \sum_{k=0}^{\infty} P[\mathbf{W}_0 \in B,\ k \leq |\xi_0| < k+1]^{\alpha} \\
&\quad \leq 2C_1' \sum_{k=0}^{\infty} \left(\frac{1}{1+k}\right)^{\alpha\beta} \left[\int_{\mathbf{W}_0 \in B,\ k \leq |\xi_0| < k+1} (1+|\xi_0|)^{\beta} dP\right]^{\alpha}, \\
&\quad \leq 2C_1' \left[\sum_{k=0}^{\infty} \left(\frac{1}{1+k}\right)^{1+\alpha}\right]^{1-\alpha} \times \left[\sum_{k=0}^{\infty} \int_{\mathbf{W}_0 \in B,\ k \leq |\xi_0| < k+1} (1+|\xi_0|)^{\beta} dP\right]^{\alpha}
\end{aligned}$$

which has the form of the right side of Eq. (13). ◇

A RENEWAL THEOREM

For the proof of Eq. (2), let

$$\begin{aligned}
\xi^{(k)}(w_1, \ldots, w_k) &= E[\xi(W_n, W_{n-1}, \ldots) | W_n = w_1, \ldots, W_{n-k+1} = w_k] \\
&= \int_{\mathcal{W}^{\mathbb{N}}} \xi(w_1, \ldots, w_k, y) Q^{\infty}\{dy\},
\end{aligned}$$

and

$$\xi_n^{(k)} = \xi^{(k)}(W_n, \ldots, W_{n-k+1}),$$

Then $E|\xi_n^{(k)} - \xi_n|$ does not depend on n and approaches 0 as $k \to \infty$, by the Martingale Convergence Theorem.

Theorem 1. *Let* $0 < \eta < 1$ *and let* m *be as in Eq.* (9). *Then*

$$\lim_{a\to\infty}\left|\sum_{n=0}^{\infty} P[(W_n,\ldots,W_{n-m+1}) \in B,\ \xi_n \le c,\ a < Z_n \le a+b] - \frac{b}{\mu} P[(W_0,\ldots,W_{-m+1}) \in B,\ \xi_0 \le c]\right| = 0, \tag{15}$$

uniformly with respect to Borel sets $B \subseteq \mathcal{W}^m$ *for all* $0 < b < \infty$ *and all continuity points* c *of* G.

Proof. It is first shown that Eq. (15) holds with Z_n replaced by S_n; Eq. (15) itself is then deduced from the Continuous Mapping Theorem. When Z_n is replaced by S_n, denote the sum on the first line of Eq. (15) by $\Sigma_a(B, c)$. Fix $b > 0$ and a continuity point c of G; let $\varepsilon > 0$ be a value for which $c \pm \epsilon$ are continuity points of G; and let

$$B^{\pm} = B \cap \{w \in \mathcal{W}^m : \xi^{(m)}(w_1,\ldots,w_m) \le c \pm \epsilon\},$$
$$\Sigma_a^{\pm} = \sum_{n=0}^{\infty} P[(W_n,\ldots,W_{n-m+1}) \in B^{\pm},\ a < S_n \le a+b],$$

and

$$r_a = \sum_{n=0}^{\infty} P[|\xi_n^{(m)} - \xi_n| \ge \epsilon,\ a < S_n \le a+b].$$

Then $\Sigma_a^{-} - r_a \le \Sigma_a(B,c) \le \Sigma_a^{+} + r_a$. So,

$$\left|\Sigma_a(B,c) - \frac{b}{\mu} P[(W_0,\ldots,W_{-m+1}) \in B,\ \xi_0 \le c]\right|$$
$$\le \max_{\pm}\left|\Sigma_a^{\pm} - \frac{b}{\mu} P[(W_0,\ldots,W_{-m+1}) \in B^{\pm}]\right| + r_a + \frac{b}{\mu} P[c - \epsilon \le \xi_0^{(m)} \le c + \epsilon].$$

By Eq. (12) with $\alpha = 1/2$,

$$|r_a| \le C_1 \sqrt{P\{|\xi_0^{(m)} - \xi_0| \ge \epsilon\}}$$

which approaches zero as $a \to \infty$. Relation (15), with Z_n replaced by S_n, then follows from Lemma 2 by letting $a \to \infty$ and $\epsilon \downarrow 0$.

Some simplifications are possible in the general case. First, it suffices to establish Eq. (15) for fixed, but arbitrary $0 < b \leq \mu$ and continuity point c of G; and it suffices to show that Eq. (15) holds for arbitrary families $B_a \subseteq \mathcal{W}^m$ (since B_a may be chosen to nearly attain the supremum). Further, it suffices to establish Eq. (15) for arbitrary sequences $A_0 = \{a_1, a_2, \ldots\}$ for which $a_k \to \infty$, and, by considering subsequences, there is no loss of generality in supposing that $\ell = \lim_{a \in A_0} P\{(W_1, \ldots, W_m) \in B_a\}$ exists. If $\ell = 0$, then Eq. (15) is clear. For then $P[(W_0, \ldots, W_{-m+1}) \in B_a] \to 0$ as $a \to \infty$ (along the subsequence), and the sum on its left side is at most

$$C_2 \left[\int_{(W_0, -, W_{-m+1}) \in B_a} (1 + |\xi_0|^{7/12}) dP \right]^{3/4},$$

by Eq. (13) with $\alpha = 3/4$. So, it suffices to consider the case $\ell > 0$.

Supposing that $\ell > 0$, let $\tilde{G}_a(c) = P\{\xi_0 \leq c | (W_0, \ldots, W_{-m+1}) \in B_a\}$. Then $\tilde{G}_a$ are tight, since $\ell > 0$, and there is no loss of generality in supposing that $\tilde{G}_a$ converges to a limit $\tilde{G}$ as $a \to \infty$ along the subsequence. Let

$$J_a(b,c) = \sum_{n=0}^{\infty} P\{(W_n, \ldots, W_{n-m+1}) \in B_a,\ \xi_n \leq c,\ a < S_n \leq a + b\},$$

$$K_a(b,c) = \sum_{n=0}^{\infty} P\{(W_n, \ldots, W_{n-m+1}) \in B_a,\ \xi_n \leq c,\ a < Z_n \leq a + b\},$$

and

$$J_\infty(b,c) = \ell \frac{b}{\mu} \tilde{G}(c)$$

for $0 \leq b \leq \mu$ and $-\infty < c < \infty$; and use the same symbols to denote the induced measures over $[0, \mu] \times \mathbb{R}$. Thus, $J_\infty = l\lambda \times \tilde{G}/\mu$ where λ is Lebesgue measure. Then J_a and K_a are finite measures over $[0, \mu] \times \mathbb{R}$ for which J_a converges weakly to J_∞, by the special case. Let $\psi(x, y) = (x + y, y)$ for $0 \leq x \leq \mu$ and $-\infty < y < \infty$, where addition is understood modulo μ. Then ψ is continuous a.e. (J_∞) and $K_a(b, c) = J_a\{(x, y) : x + y \leq b,\ y \leq c\}$; that is, $K_a = J_a \circ \psi^{-1}$. Clearly, $J_\infty \circ \psi^{-1} = J_\infty$, by the translation in variance of Lebesgue measure. That K_a converges weakly to $J_\infty = J_\infty \circ \psi^{-1}$ now follows directly from the Continuous Mapping Theorem to complete the proof. $\diamond$

THE EXCESS

The Limiting Distribution. Recall that the first passage times t_a and excesses R_a are defined by Eqs. (3) and (4), and observe that

$$\lim_{a\to\infty} \frac{1}{a} t_a = \frac{1}{\mu} w.p.1,$$

since $\lim_{n\to\infty} \xi_n/n = 0\, w.p.1$, $\lim_{n\to\infty} Z_n/n = \mu \quad w.p.1$, and $Z_{t_a-1}/t_a \leq a/t_a \leq Z_{t_a}/t_a$ for all sufficiently large a,

Theorem 2. *For $0 < r < s < \infty$ and continuity points $-\infty < b < c < \infty$ of G,*

$$\lim_{a\to\infty} P[b < \xi_{t_a} \leq c, r < R_a \leq s] = \frac{1}{\mu}\int_r^s P[b < \xi_0 \leq c, \inf_{k<0} Z_k \geq u]\, du,$$

where $Z_n = (X_{n+1} + \cdots + X_0) + (\xi_0 - \xi_n)$ for $n \leq -1$.

Proof. Let $m = m_a$ be as in Eq. (9) and let $k = k_a = m/2$. Then

$$P[t_a \geq m, r < R_a \leq s, b < \xi_{t_a} \leq c] = \sum_{n=m}^{\infty} P[t_a \geq n, r < Z_n - a \leq s, b < \xi_n \leq c]$$

If $\epsilon > 0$, then

$$\begin{aligned}\{t_a \geq n, r < Z_n - a \leq s\} &= \{Z_j \leq a, \forall 1 \leq j < n, r < Z_n - a \leq s\}\\ &\subseteq \{Z_n - Z_{n-j} \geq r, \forall 1 < j \leq k\}\\ &\subseteq \{S_n - S_{n-j} + (\xi_n^{(k)} - \xi_{n-j}^{(k)}) \geq r - 2\epsilon, \forall 1 \leq j < k\}\\ &\cup \{|\xi_{n-j} - \xi_{n-j}^{(k)}| \geq \epsilon, \exists 1 \leq j < k\}\end{aligned}$$

for all $n \geq m$, where $\forall$ and $\exists$ are to be read "for all" and "for some". Let B_a be the set of $w = (w_0, \ldots, w_{-m+1}) \in \mathcal{W}^m$ for which

$$x_0 + \cdots + x_{-j+1} + \xi^{(k)}(w_0, \ldots, w_{-k+1}) - \xi^{(k)}(w_{-j}, \ldots, w_{-j-k+1}) \geq r - 2\epsilon$$

for all $1 \leq j < k$, where $x_j = \varphi(w_j)$; and let

$$C_a = \{w \in \mathcal{W}^N : |\xi(w_{-j}, w_{-j-1}, \ldots) - \xi^{(k)}(w_{-j}, \ldots, w_{-j-k+1})| \geq \epsilon \,\exists 1 \leq j < k\}.$$

Then $\{t_a \geq n, r < Z_n - a\} \subseteq \{(W_n, \ldots, W_{n-m+1}) \in B_a\} \cup \{\mathbf{W}_n \in C_a\}$. So,

$$\begin{aligned}
&\sum_{n=m}^{\infty} P[t_a \geq n, r < Z_n - a \leq s, b < \xi_n^k \leq c] \\
&\quad \leq \sum_{n=m}^{\infty} P[(W_n, \ldots, W_{n-m+1}) \in B_a, r < Z_n - a \leq s, b < \xi_n^{(k)} \leq c] \\
&\quad + \sum_{n=m}^{\infty} P[\mathbf{W}_n \in C_a, r - c < S_n - a \leq s - b]. \\
&\quad \leq \frac{s-r}{\mu} P[(W_0, \ldots, W_{-m+1}) \in B_a, b < \xi_0^{(k)} \leq c] \\
&\quad + o(1) + C_1 \sqrt{P[\mathbf{W}_0 \in C_a]},
\end{aligned}$$

So, since $P[t_a < m] \to 0$, and $P[(W_0, \ldots, \mathbf{W}_{-m+1}) \in B_a] \to P[\inf_{k<0} Z_k \geq r - 2\epsilon]$ as $a \to \infty$, it follows from Theorem 1 that

$$\limsup_{a\to\infty} P[r < R_a \leq s, b < \xi_{t_a} \leq c] \leq \frac{s-r}{\mu} P[b < \xi_0 \leq c, \inf_{k<0} Z_k \geq r], \tag{16}$$

by first letting $a \to \infty$ and then $\epsilon \downarrow 0$. Next, partition the interval $(r, s]$ into ℓ subintervals of equal length as $r = r_0 < r_1 < \cdots < r_\ell, = s$, so that $P[r < R_a < s, b \leq \xi_{t_a} \leq c] = \sum_{j=1}^{\ell} P[r_{j-1} < R_a \leq r_j, b < \xi_{t_a} \leq c]$. Applying (16) to each term in the sum and then letting $\ell \to \infty$, it follows that

$$\limsup_{a\to\infty} P[b < \xi_{t_a} \leq c, r < R_a \leq s] \leq \frac{1}{\mu} \int_r^s P[b < \xi_0 \leq c, \inf_{k<0} Z_k \geq u]\, du.$$

A dual lower may be obtained similarly to complete the proof. $\diamond$

Corollary. *As* $a \to \infty$, $(R_a, \xi_{t_a}) \Rightarrow H$, *where* H *is as in Eq.* (5) *and* $\Rightarrow$ *denotes convergence in distribution.*

Proof. This is clear. $\diamond$

Uniform Integrability. Let $U_n = X_n + \xi_n - \xi_{n-1}$, so that $Z_n = Z_{n-1} + U_n$.

Lemma 5. *For any* $0 < \alpha < 1$, *there is a constant* C_3 *for which*

$$P\{\mathbf{W}_{t_a} \in B\} \leq C_3 \left[\int_{\mathbf{W}_0 \in B} (1 + U_0^+)^{1+\beta} (1 + |\xi_0|)^\beta \, dP \right]^\alpha$$

for all Borel sets $B \subseteq \mathcal{W}^N$ *and* $a > 0$, *where* $\beta = (1 - \alpha^2)/\alpha$.

Proof. Since $\{t_a = n\} \subseteq \{Z_{n-1} \le a < Z_n\}$,

$$\begin{aligned} P\{\mathbf{W}_{t_a} \in B\} &\le \sum_{n=1}^{\infty} P\{Z_{n-1} \le a < Z_n, \mathbf{W}_n \in B\} \\ &\le \sum_{k \le a+1} \sum_{n=1}^{\infty} P\{k-1 < Z_{n-1} \le k, U_n \ge a-k, \mathbf{W}_n \in B\} \\ &\le C_2 \sum_{k \le a+1} \left[\int_{\mathbf{W}_0 \in B,\, U_0 > a-k} (1+|\xi_0|)^\beta \, dP \right]^\alpha, \end{aligned}$$

by Eq. (13). As in proof of Lemma 4, the last line is at most

$$\begin{aligned} &2C_2 \sum_{k=0}^{\infty} \left(\frac{1}{1+k}\right)^{\alpha\beta} \left[\int_{\mathbf{W}_0 \in B,\, U_0 > k} (1+U_0^+)^\beta (1+|\xi_0|)^\beta \, dP \right]^\alpha \\ &\le 2C_2 \left[\sum_{k=0}^{\infty} \left(\frac{1}{1+k}\right)^{1+\alpha} \right]^{1-\alpha} \times \left[\sum_{k=0}^{\infty} \int_{\mathbf{W}_0 \in B,\, U_0 > k} (1+U_0^+)^\beta (1+|\xi_0|)^\beta \, dP \right]^\alpha \end{aligned}$$

Finally, reversing the orders of summation and integration, in the second sum on the last line is at most

$$\int_{\mathbf{W}_0 \in B} \sum_{k \le U_0} (1+U_0^+)^\beta (1+|\xi_0|)^\beta \, dP \le \int_{\mathbf{W}_0 \in B} (1+U_0^+)^{1+\beta} (1+|\xi_0|)^\beta \, dP,$$

completing the proof of the lemma. ◇

Theorem 3. *If F and G have finite $(2+\delta)$th moments for some $\delta > 0$, then R_a and ξ_{t_a} are uniformly integrable in $a > 0$.*

Proof. Suppose that $0 < \delta \le 1$ (without loss of generality); recall that $\beta = (1-\alpha^2)/\alpha$; let $\gamma = (2-\alpha)/\alpha$; let $\alpha < 1$ be so close to one that $\beta \le \delta/4$ and $\gamma \le 1 + \delta/4$. Then $E[(1+U_0^+)^{1+\beta}(1+|\xi_0|)^{\beta+\gamma}] < \infty$ by Schwarz' Inequality and $E[(1+U_0^+)^{1+\beta+\gamma}(1+|\xi_0|)^\beta] < \infty$ by Hölder's Inequality. It follows that

$$\begin{aligned} P\{|\xi_{t_a}| > r\} &\le C_3 \left[\int_{|\xi_0| > r} (1+U_0^+)^{1+\beta} (1+|\xi_0|)^\beta \, dP \right]^\alpha, \\ &\le \frac{C_3}{(1+r)^{\alpha\gamma}} \left[\int (1+U_0^+)^{1+\beta} (1+|\xi_0|)^{\gamma+\beta} \, dP \right]^\alpha, \end{aligned}$$

which is independent of a and integrable over $0 \le r < \infty$, since $\alpha\gamma = 2 - \alpha > 1$. So ξ_{t_a} are uniformly integrable. Similarly,

$$P[R_a > r] \leq P[U_{t_a}^+ > r] \leq C_3 \left[\int_{U_0^+ > r} (1+U_0^+)^{1+\beta} (1+|\xi_0|)^\beta \, dP \right]^\alpha$$
$$\leq \frac{C_3}{(1+r)^{\alpha\gamma}} \left[\int_{U_0^+ > r} (1+U_0^+)^{1+\beta+\gamma} (1+|\xi_0|)^\beta \, dP \right]^\alpha,$$

so that R_a are uniformly integrable.

Corollary. *Let ρ and v be the means of the asymptotic distributions of R_a and ξ_{t_a}. Then*

$$E(t_a) = \frac{1}{\mu}(a + \rho - v) + o(1).$$

Proof. This follows from Wald's Lemma, Theorem 3, and the relation $S_{t_a} + \xi_{t_a} = a + R_a$. ◇

Approximate stationarity. In the application to the SPRT, the log-likelihood ratios were of the form

$$\tilde{Z}_n = S_n + \tilde{\xi}_n,$$

where S_n is a random walk with a positive drift and $\tilde{\xi}_n$ were approximately stationary. Consider $\tilde{\xi}_n$ of the form

$$\tilde{\xi}_n = \zeta_n(\mathbf{W}_n), \tag{17}$$

where $\zeta_n : \mathcal{W} \to \mathbb{R}$ are measurable functions for which $\lim_{n\to\infty} \zeta_n(w) = \xi(w)$ for *a.e.* $w(Q^\infty)$; and let

$$\tilde{t}_a = \inf\{n \geq 1 : \tilde{Z}_n > a\} \tag{17a}$$

and

$$\tilde{R}_a = \tilde{Z}_{\tilde{t}_a} - a. \tag{17b}$$

The next theorem provides conditions under which $\tilde{R}_a$ and $\tilde{\xi}_{\tilde{t}_a}$ have the same asymptotic properties as R_a and ξ_{t_a}. Let

$$\bar{\zeta}(w) = \sup_n |\zeta_n(w)|,$$
$$\bar{\zeta}_n = \bar{\zeta}(\mathbf{W}_n),$$
$$V_n = X_n^+ + \bar{\zeta}_n + \bar{\zeta}_{n-1}.$$

Lemma 6. *For any* $0 < b < \infty$ *and* $0 < \alpha < 1$, *there are constant* C_3 *and* C_4 *for which*

$$\sum_{n=0}^{\infty} P[\mathbf{W}_n \in B, a < \tilde{Z}_n \le a+b] \le C_3\left[\int_B (1+\bar{\zeta}_0)^{1+\beta}\, dP\right]^{\alpha} \tag{18}$$

and

$$P[\mathbf{W}_{\tilde{t}_a} \in B] \le C_4\left[\int_{V_0\in B} (1+V_0)^{1+\beta}(1+\bar{\zeta}_0)^{1+\beta}\, dP\right]^{\alpha} \tag{19}$$

for all Borel sets $B \subseteq \mathcal{W}^{\mathbb{N}}$, *where* $\beta = (1-\alpha^2)/\alpha$.

Proof. The proof is similar to those of Lemmas 4 and 5, effectively trading additional moment conditions for the exact stationarity. The details are presented after the proof of Theorem 4.

Theorem 4. *Let* ξ_n *be as in Eq.* (8) *and* $\tilde{\xi}_n, n \ge 1$, *be as in Eq.* (17). *If* $\lim_{n\to\infty} |\tilde{\xi}_n - \xi_n| = 0$ *w.p.1, then*

$$\lim_{a\to\infty} P[\tilde{t}_a = t_a] = 1 \tag{21}$$

and

$$|\tilde{\xi}_{\tilde{t}_a} - \xi_{t_a}| + |\tilde{R}_a - R_a| \to 0 \tag{22}$$

in probability as $a \to \infty$. *If also,* X_0 *and* $\bar{\zeta}_0$ *have finite* $(3+\delta)$*th moments for some* $\delta > 0$, *then* $\tilde{R}_a$ *and* $\tilde{\xi}_{\tilde{t}_a}$ *are uniformly integrable.*

Proof. First observe that $\lim_{n\to\infty}\tilde{\xi}_n/n \to 0$ *w.p.*1 and, therefore, $\lim_{a\to\infty} \tilde{t}_a/a = 1/\mu$. Next, let $\epsilon > 0$ and $0 < \eta < 1$, and let B_a be the event

$$B_a = \left\{t_a \ge m,\ \tilde{t}_a \ge m,\ R_{a-\epsilon} \ge 2\epsilon,\ \sup_{n\ge m} |\tilde{\xi}_n - \xi_n| \le \epsilon\right\},$$

where m is as in Eq. (9). Then B_a implies that $t_a = t_{a-\epsilon} = \tilde{t}_a$. The first equality is clear. For the second, B_a implies that $\tilde{t}_a \ge m$, that $\tilde{Z}_k = Z_k + \tilde{\xi}_k - \xi_k \le a$ for $m \le k < t_{a-\epsilon}$, and that $\tilde{Z}_{t_a} = Z_{t_a} + \tilde{\xi}_{t_a} - \xi_{t_a} > a + \epsilon - \epsilon = a$, so that $\tilde{t}_a = t_a$. Thus,

$$\limsup_{a\to\infty} P\big[\tilde{t}_a \ne t_a\big] \le \lim_{a\to\infty} P(B_a') = H(2\epsilon).$$

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Relation (21) follows since $\lim_{\epsilon\to 0} H(2\epsilon) = 0$; and Eq. (22) is an easy consequence. For example, $|\tilde{\xi}_{\tilde{t}_a} - \xi_{t_a}| \leq |\tilde{\xi}_{\tilde{t}_a} - \xi_{\tilde{t}_a}| + |\xi_{\tilde{t}_a} - \xi_{t_a}|$ of which the first term approaches zero $w.p.1$ as $a \to \infty$ of the second equals zero with probability approaching one.

The proof the uniform integrability is similar to that in Theorem 3. For $\tilde{\xi}_{\tilde{t}_a}$, let $\alpha < 1$ be so close to one $6\beta < \delta$ and $\gamma < 1 + (1/2)\delta$. Then $(1 + V_0)^{1+\beta}(1 + \bar{\zeta}_0)^{1+\gamma+\beta}$ is integrable by Hölder's Inequality, and

$$\begin{aligned} P\Big[\tilde{\xi}_{\tilde{t}_a} > r\Big] &\leq P\big[\bar{\zeta}_{\tilde{t}_a} > r\big] \\ &\leq C_3\left[\int_{\bar{\zeta}_0 > r} (1 + V_0)^{1+\beta}(1 + \bar{\zeta}_0)^{1+\beta} dP\right]^{\alpha} \\ &\leq \frac{C_3}{(1+r)^{\alpha\gamma}}\left[\int_{\bar{\zeta}_0 > r} (1 + V_0)^{1+\beta}(1 + \bar{\zeta}_0)^{1+\gamma+\beta} dP\right]^{\alpha}, \end{aligned}$$

which is independent of a and integrable of $0 \leq r < \infty$. So, $\tilde{\xi}_{\tilde{t}_a}$ are uniformly integrable; and $\tilde{R}_a$ may be handled similarly. $\diamond$

Proof of Lemma 6. Let $A_k = \{w \in \mathcal{W}^{\mathbb{N}} : \inf_j |\zeta_j(w) - k| \leq 1/2\}$ for $k = 0, \pm 1, \pm 2, \ldots$. Then the left side of Eq. (18) is at most

$$\begin{aligned} &\sum_{k=-\infty}^{\infty}\sum_{n=0}^{\infty} P\Big[\mathbf{W}_n \in B, k - \tfrac{1}{2} < \tilde{\xi}_n \leq k + \tfrac{1}{2}, a - k - \tfrac{1}{2} < S_n \leq a + b - k + \tfrac{1}{2}\Big] \\ &\quad \leq \sum_{k=-\infty}^{\infty}\sum_{n=0}^{\infty} P\big[\mathbf{W}_n \in A_k \cap B, a - k - \tfrac{1}{2} < S_n \leq a + b - k + \tfrac{1}{2}\big] \\ &\quad \leq C_2 \sum_{k=-\infty}^{\infty} P[\mathbf{W}_n \in A_k \cap B]^d \end{aligned}$$

for some constant C_2, by Eq. (12). If $k \neq 0$, then $A_k \subseteq \{w : \bar{\zeta}(w) \geq \frac{1}{2}|k|\}$, and

$$P[\mathbf{W}_0 \in A_k \cap B] \leq \left(\frac{2}{1 + |k|}\right)^{\beta} \int_{\mathbf{W}_o \in B,\ \bar{\zeta}_0 > \frac{1}{2}|k|} (1 + \bar{\zeta}_0)^{\beta} dP;$$

and the latter inequality is also (trivially) true if $k = 0$. So, summing over k and using Hölder's Inequality with $p = 1/(1 - \alpha)$ and $q = 1/\alpha$,

$$\sum_{k=-\infty}^{\infty} P[\mathbf{W}_n \in A_k \cap B]^{\alpha}$$

$$\leq 2\sum_{k=0}^{\infty}\left(\frac{2}{1+k}\right)^{\alpha\beta}\left[\int_{\mathbf{W}_o \in B, \bar{\zeta}_o > \frac{1}{2}k} (1+\bar{\zeta}_0)^{\beta} dP\right]^{\alpha}$$

$$\leq \left[2\sum_{k=0}^{\infty}\left(\frac{2}{1+k}\right)^{1+\alpha}\right]^{1-\alpha}\left[\sum_{k=-\infty}^{\infty}\int_{\mathbf{W}_o \in B, \bar{\zeta}_o > \frac{1}{2}k} (1+\bar{\zeta}_0)^{\beta} dP\right]^{\alpha}$$

from which Eq. (18) follows by reversing the summation and integration symbols in the second factor.

For Eq. (19), first observe that $\tilde{Z}_n - \tilde{Z}_{n-1} = X_n + \tilde{\xi}_n - \tilde{\xi}_{n-1} \leq X_n^+ + \bar{\zeta}_n + \bar{\zeta}_{n-1} = V_n$ for all n. So, there is a constant C_3 for which

$$P[\mathbf{W}_{\tilde{t}_a} \in B] \leq \sum_{n=1}^{\infty} P[\tilde{Z}_{n-1} \leq a < \tilde{Z}_n, \mathbf{W}_n \in B]$$

$$\leq \sum_{k \leq a+1}\sum_{n=1}^{\infty} P[k-1 < \tilde{Z}_{n-1} \leq k,\ V_n > a-k,\ \mathbf{W}_n \in B]$$

$$\leq C_3 \sum_{k \leq a+1}\left[\int_{\mathbf{W}_0 \in B,\ V_0 > a-k} (1+\bar{\zeta}_0)^{1+\beta} dP\right]^{\alpha}.$$

As above, the last line is at most

$$2C_3\sum_{k=0}^{\infty}\left(\frac{1}{1+k}\right)^{\alpha\beta}\left[\int_{\mathbf{W}_0 \in B,\ V_0 \geq k} (1+V_0)^{\beta}(1+\bar{\zeta}_0)^{1+\beta} dP\right]^{\alpha}$$

$$\leq 2C_3\left[\sum_{k=0}^{\infty}\left(\frac{1}{1+k}\right)^{\alpha\beta}\right]^{1-\alpha}\left[\sum_{k=0}^{\infty}\int_{\mathbf{W}_o \in B,\ V_0 \geq k} (1+V_0)^{\beta}(1+\bar{\zeta}_0)^{1+\beta} dP\right]^{\alpha}$$

$$\leq 2C_3\left[\sum_{k=0}^{\infty}\left(\frac{1}{1+k}\right)^{\alpha\beta}\right]^{1-\alpha}\left[\int_{\mathbf{W}_o \in B} (1+V_0)^{1+\beta}(1+\bar{\zeta}_0)^{1+\beta} dP\right]^{\alpha},$$

establishing Eq. (19).

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