An example of non-quenched convergence in the conditional central limit theorem for partial sums of a linear process

Dalibor Volný and Michael Woodroofe

Abstract A causal linear processes \( \cdots X_{-1}, X_0, X_1, \cdots \) is constructed for which the conditional distributions of standardized partial sums \( S_n = X_1 + \cdots + X_n \) given \( \cdots X_{-1}, X_0 \) converge in probability to the standard normal distribution, but do not converge w.p. 1.

1 Introduction

Let \( \cdots X_{-1}, X_0, X_1, \cdots \) denote a strictly stationary sequence defined on a probability space \( (\Omega, \mathcal{A}, P) \) and adapted to a filtration \( \mathcal{F}_k \). Suppose that the \( X_k \) have mean 0 and finite variance; and let \( S_n = X_1 + \cdots + X_n \) and \( \sigma_n^2 = E(S_n^2) \). With this notation the Conditional Central Limit Question may be stated: When do the conditional distributions of a \( S_n/\sigma_n \) converge in probability to the standard normal distribution; that is, when does the Levy distance between the conditional distribution and the standard normal converge in probability zero? Two sets of necessary and sufficient conditions may be found in [3] and [10].

One can also ask: When is the convergence quasiomed; that is, when do the conditional distributions converge almost surely? In a Markov Chain setting, this means that the convergence takes place for almost every (with respect to the stationary measure) starting point. It has been shown that several important classical limit theorems are quenched. See [1] and (for more recent research, e.g.) [5], [2], and [11].
For a causal linear process

$$X_k = \sum_{i=0}^{\infty} a_i \xi_{k-i},$$

(1)

where $a_0, a_1, \ldots$ are square summable and $\xi_{-1}, \xi_0, \xi_1, \ldots$ are i.i.d. with mean 0 and variance one, let $\mathcal{F}_n = \sigma(\ldots, \xi_{n-1}, \xi_n)$. Then

$$S_n = \sum_{i=0}^{n} [b_{i+n} - b_i] \xi_{i-n} + \sum_{i=1}^{n} b_{n-i} \xi_i,$$

where $b_n = a_0 + \cdots + a_n$. It follows that $E(S_n|\mathcal{F}_0) = \sum_{i=0}^{n} [b_{i+n} - b_i] \xi_{i-n}$ and $S_n - E(S_n|\mathcal{F}_0) = \sum_{i=1}^{n} b_{n-i} \xi_{i-n}$ are independent. Then, letting $\| \cdot \|$ denote the norm in $L^2(P)$ and $\tau_n = b_0^2 + \cdots + b_{n-1}^2$,

$$\|E(S_n|\mathcal{F}_0)\|^2 = \sum_{i=0}^{n} [b_{i+n} - b_i]^2 = \nu_n^2 \quad \text{say},$$

and

$$\sigma_n^2 = E [E(S_n|\mathcal{F}_0)^2] + E [E(S_n - E(S_n|\mathcal{F}_0))^2] = \nu_n^2 + \tau_n^2.$$  

With this notation, Wu and Woodroofe [10] showed that if $\lim_{n \to \infty} \sigma_n^2 = \infty$, then the conditional distribution function of $S_n/\sigma_n$ given $\mathcal{F}_0$ converges in probability to the standard normal distribution iff $\nu_n^2 = o(\sigma_n^2)$. Here a causal linear process with absolutely summable coefficients is constructed for which the conditional distributions of $S_n/\sigma_n$ converge to $\Phi$ in probability but not with probability one. The summability of coefficients means that Hannan’s condition [7], $\sum_{i \geq 0} \|E(X_0|\mathcal{F}_i) - E(X_0|\mathcal{F}_{i-1})\| < \infty$, is satisfied and, therefore, that the (weak) invariance principle holds (see [4]).

## 2 The preliminaries

The first step is to develop a necessary and sufficient condition for quenched convergence. The proof of the lemma below uses the Convergence of Types Theorem ([8], p. 203): Let $Y_n$ and $Z_n$ be random variables of the form $Y_n = \alpha_n Z_n + \beta_n$, where $\alpha_n$, $\beta_n \in \mathbb{R}$ are constants. If $Y_n \Rightarrow Y$ and $Z_n \Rightarrow Z$, where $Z$ non-degenerate, then $\alpha_n$ and $\beta_n$ converge to limits $\alpha$ and $\beta$ and $Y = \text{dist} \alpha Z + \beta$.

**Lemma 1** For a causal linear process (1) for which $\sigma_n \to \infty$ and $\nu_n = o(\sigma_n)$, the conditional distribution of $S_n/\sigma_n$ given $\mathcal{F}_0$ converges to the standard normal distribution w.p. 1 iff

$$\lim_{n \to \infty} \frac{1}{\sigma_n} E(S_n|\mathcal{F}_0) = 0 \text{ w.p. 1}. \quad (2)$$

**Proof.** Suppose first that $\nu_n = o(\sigma_n)$ in which case $\tau_n^2/\sigma_n^2 \to 1$, and let $F_n$ denote the (unconditional) distribution function of $[S_n - E(S_n|\mathcal{F}_0)]/\tau_n$. Then

$$P \left[ \frac{S_n}{\sigma_n} \leq z | \mathcal{F}_0 \right] = F_n \left[ \frac{\sigma_n z - E(S_n|\mathcal{F}_0)}{\tau_n} \right],$$

where $\sigma_n \to \infty$. Therefore, if $\nu_n = o(\sigma_n)$, then $P \left[ \frac{S_n}{\sigma_n} \leq z | \mathcal{F}_0 \right] \to \Phi(z)$, from which the lemma follows. If $\nu_n = \omega(\sigma_n)$, then $\|S_n - E(S_n|\mathcal{F}_0)\|/\sigma_n \to \infty$, and $P \left[ \frac{S_n}{\sigma_n} \leq z | \mathcal{F}_0 \right] \to \Phi(z)$ in probability. Thus, the lemma is proved.
by independence. It is shown below that $F_n$ converges weakly to the standard normal distribution $\Phi$. The sufficiency of (2) for almost sure convergence of the conditional distribution function is then obvious. Conversely, if the conditional distributions converge almost surely to the standard normal distribution, then $E(S_n|F_0)/\sigma_n \to 0$ w.p. 1, by the Convergence of Types Theorem, applied conditionally.

That $F_n \Rightarrow \Phi$ follows from Theorem 2.1 and Corollary 2.1 of [9]. The argument is sketched here for completeness. By the Lindeberg–Feller Condition, it is sufficient to show that

$$\lim_{n \to \infty} \max_{0 \leq i \leq n} \frac{b_i^2}{\tau_n^2} = 0. \quad (3)$$

Suppose that the maximum is attained at $i_n$ and (temporarily) that $i_n \leq \frac{1}{2}n$; and let $A^2 = \sum_{i=0}^{\infty} a_i^2$. If $k \leq \frac{1}{2}n$, then

$$b_{i_n}^2 - b_{i_n+k}^2 = \sum_{j=1}^{k} (b_{i_n+j-1} - b_{i_n+j})(b_{i_n+j-1} + b_{i_n+j}) \leq \sum_{j=1}^{k} a_{i_n+j} \left( \sum_{j=1}^{k} (b_{i_n+j-1} + b_{i_n+j})^2 \right) \leq 2A\tau_n.$$

So, for any $m \leq \frac{1}{2}n$, $mb_{i_n}^2 \leq \sum_{k=1}^{m} b_{i_n+k}^2 + 2Am\tau_n \leq \tau_n^2 + 2Am\tau_n$ and, therefore

$$\frac{b_{i_n}^2}{\tau_n} \leq \frac{1}{m} + \frac{2A}{\tau_n}.$$

The same inequality may be obtained if $i_n \geq \frac{1}{2}n$ by a dual argument in which $k$ is replaced by $-k$, and (3) follows by letting $n \to \infty$ and $m \to \infty$ in that order. \hfill $\Box$

**Lemma 2** For a causal linear process (1): If $a_0, a_1, a_2, \cdots$ are absolutely summable and $b := \sum_{i=0}^{\infty} a_i \neq 0$, then $\sigma_n^2 \sim b^2n$ and $\nu_n^2 = o(\sigma_n^2)$.

**Proof.** The proof uses a different expression for $E(S_n|F_0)$,

$$E(S_n|F_0) = \sum_{j=1}^{n} a_j \xi_j + \sum_{j=n+1}^{\infty} a_j[\xi_j - \xi_j-n],$$

where $\xi_j = \xi_{j+1} + \cdots + \xi_0$. Thus, $\tau_n^2 = b_0^2 + \cdots + b_{n-1}^2 \sim b^2n$ and

$$\|E(S_n|F_0)\| \leq \sum_{j=1}^{n} a_j \sqrt{j} + \sum_{j=n+1}^{\infty} a_j \sqrt{n} = o(\sqrt{n}) = o(\sigma_n).$$

The lemma follows directly. \hfill $\Box$
3 The example

The main result follows.

**Theorem 1** There are non-negative summable coefficients $a_0, a_1, a_2, \cdots$ for which $\nu_n^2 = o(\sigma_n^2)$ but (2) fails.

**Proof.** By Lemma 2, it suffices to construct positive summable coefficients $a_0, a_1, a_2, \cdots$ for which (2) fails. We consider coefficients of the form

$$a_n = \frac{1}{2j\sqrt{n-j-1}},$$

for $j \geq 1$, and $a_i = 0$ if $i \notin \{n_1, n_2, \cdots\}$, where $0 = n_0 < n_1 < n_2 < \cdots$ is a sequence of positive integers constructed below. It is clear that a sequence of the form (4) is summable.

For sequences of the form (4),

$$E(S_n | F_0) = \sum_{1 \leq n \leq n_k} a_n \zeta_{n_k} + \sum_{n_k < n} a_n [\zeta_n - \zeta_{n_k-n}],$$

where $\zeta_n = \xi_{n+1} + \cdots + \xi_0$, as above. So, for $n_{k-1} \leq n < n_k$

$$E(S_n | F_0) = I_k(n) + II_k(n),$$

where

$$I_k(n) = \sum_{j=1}^{k-1} a_n \zeta_{nj} + a_n [\zeta_{n_k} - \zeta_{nj-n}],$$

$$II_k(n) = \sum_{j=k+1}^{\infty} a_n [\zeta_{nj} - \zeta_{nj-n}],$$

and the empty sum is to be interpreted as zero when $k = 1$.

The specification of the integers $n_i$ depends on the following claim: There are integers $0 < n_1 < n_2 < \cdots$ for which

$$P\left[ \max_{n_{k-1} \leq n < n_k} \frac{I_k(n)}{\sqrt{n}} > 2^{k+1} \right] > 1 - \frac{1}{2^{k+1}},$$

for all $k \geq 1$. The proof of the claim, in turn, depends on the following observation:

Let $J(N,n) = [\zeta_N - \zeta_{N-n}] = \xi_{N+1} + \cdots + \xi_{n-N}$ for $1 \leq n \leq N$, so that $I_k(n) = \sum_{j=1}^{k-1} a_n \zeta_{nj} + a_n J(n_k,n)$ for $k \geq 1$. Then the joint distribution of $J(N,n)$, $1 \leq n \leq N$, is the same as the joint distribution of $\zeta_n = \xi_1 + \cdots + \xi_n$, $1 \leq n \leq N$. It then follows from the Law of the Iterated Logarithm that

$$P\left[ \max_{0 \leq n < N} \frac{J(N,n)}{\sqrt{n}} > 8 \right] = P\left[ \max_{0 \leq n < N} \frac{\zeta_n}{\sqrt{n}} > 8 \right] \to 1$$
as \( N \to \infty \). So the right side is at least 3/4 all sufficiently large \( N \), and the existence of \( n_1 \) follows (since \( I_1(n) = J(n_1,n)/2 \)). Now, let \( k \geq 2 \) and suppose that \( n_1, \ldots, n_{k-1} \) have been constructed. Then \( \lambda_k \) can be chosen so that

\[
P \left[ \sum_{j=1}^{k-1} a_{nj} \xi_j \right] > \lambda_k \right] \leq \left( \frac{1}{2} \right)^{k+2}.
\]

As above,

\[
P \left[ \max_{n_{k-1}\leq n<N} \frac{J(N,n)}{\sqrt{n}} > \sqrt{2^{k+1}n_k} (\lambda_k + 2^{k+1}) \right] = P \left[ \max_{n_{k-1}\leq n<N} \frac{n_k'}{\sqrt{n}} > \sqrt{2^{k+1}n_k} (\lambda_k + 2^{k+1}) \right],
\]

which approaches 1 as \( N \to \infty \) by the Law of the Iterated Logarithm. The existence of \( n_k \) in (6) follows directly from the last two displays and (5). The existence of the sequence then follows from mathematical induction (the existence of \( n_1, \ldots, n_{k-1} \) implies that of \( n_k \)).

The next step is to bound the term \( II_k(n) \). By Doob’s (1953) maximal inequality, \( E[\max_{k\leq n} |\xi_k|] \leq 2\sqrt{n} \) for all \( n \). Then

\[
E \left[ \max_{n_{k-1}\leq n<n_k} |II_k(n)| \right] \leq \sum_{j=k}^{\infty} a_{nj} E \left[ \max_{n_{k-1}\leq n<n_k} |\xi_j - \xi_{j-1}| \right] \leq \sum_{j=k}^{\infty} \frac{2\sqrt{n_j}}{2\sqrt{n_j}} \leq \frac{1}{2^{k+1}}.
\]

So

\[
P \left[ \max_{n_{k-1}\leq n<n_k} |II_k(n)| > 2^k \right] \leq \frac{1}{2^{k+1}}.
\]

That (2) fails for this construction may be seen as follows: Clearly,

\[
\max_{n_{k-1}\leq n<n_k} \frac{1}{\sqrt{n}} \sqrt{n} E(S_n | F_0) \geq \max_{n_{k-1}\leq n<n_k} \frac{I_k(n)}{\sqrt{n}} - \max_{n_{k-1}\leq n<n_k} \frac{|II_k(n)|}{\sqrt{n_k}}.
\]

So,

\[
P \left[ \max_{n_{k-1}\leq n<n_k} \frac{1}{\sqrt{n}} \sqrt{n} E(S_n | F_0) \leq 2^k \right] \leq P \left[ \max_{n_{k-1}\leq n<n_k} \frac{I_k(n)}{\sqrt{n}} \leq 2^{k+1} \right] + P \left[ \max_{n_{k-1}\leq n<n_k} |II_k(n)| \geq 2^k \right] \leq \frac{2}{2^k}
\]

for \( k \geq 2 \). It follows that

\[
P \left[ \max_{n_{k-1}\leq n<n_k} \frac{1}{\sqrt{n}} \sqrt{n} E(S_n | F_0) \leq 2^k \right], \text { infinitely often } = 0
\]

and, therefore, that \( \limsup_{n \to \infty} E(S_n | F_0) / \sqrt{n} = \infty \) w.p. 1. \( \square \)
References


