Central Limit Theorems For Superlinear Processes

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Abstract

The Central Limit Theorem is studied for stationary sequences that are sums of countable collections of linear processes. Two sets of sufficient conditions are obtained. One restricts only the coefficients and is shown to be best possible among such conditions. The other involves an interplay between the coefficients and the distribution functions of the innovations and is shown to be necessary for the Conditional Central Limit Theorem in the case of a causal process with independent innovations.

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1 Introduction

Let $\mathbb{Z}$ denote the integers, T an invertible measure preserving transformation of a probability space $(\Omega, \mathcal{A}, P)$, and $\ldots \mathcal{F}_{-1}, \mathcal{F}_0, \mathcal{F}_1, \ldots \subseteq \mathcal{A}$ a filtration for which $\mathcal{F}_{k+1} = T^{-1}\mathcal{F}_k$ for all $k \in \mathbb{Z}$. Next, let $\mathcal{F}_{-\infty}$ and $\mathcal{F}_\infty$ denote the intersection and join of the filtration, and let $g \in L^2(\Omega, \mathcal{A}, P)$ denote an $\mathcal{F}_\infty$-measurable, square integrable random variable for which $E(g|\mathcal{F}_{-\infty}) = 0$. Then $X_k := g \circ T^k,\ k \in \mathbb{Z}$, defines a strictly stationary process, sometimes called a dynamical system. Conversely, any ergodic strictly stationary sequence has a version of this form, [3], p. 107. An important special case occurs when $(\Omega, \mathcal{A}, P)$ is a power space, say $(\Omega, \mathcal{A}, P) = (\Omega_0^{\mathbb{Z}}, \mathcal{A}_0^{\mathbb{Z}}, P_0^{\mathbb{Z}})$, and T is the shift transformation $T(\cdots, \omega_{-1}, \omega_0, \omega_1, \cdots) = (\cdots, \omega_0, \omega_1, \omega_2, \cdots)$.

Theorem 1 *There are a countable set J, a square summable array $a_{i,j},\ i \in \mathbb{Z},\ j \in J$, and orthonormal random variables $\xi_{i,j},\ i \in \mathbb{Z},\ j \in J$ for which: $\xi_{i,j},\ i \in \mathbb{Z}$ are martingale differences with respect to $\mathcal{F}_i$ for each j and*

$$X_k = \sum_{j \in J} \sum_{i \in Z} a_{i,j} \xi_{k-i,j} \tag{1}$$

for each k, with the sum converging in $L^2(\Omega, \mathcal{A}, P)$.

Theorems 1 and 2 (below) are established in Section 2.

Observe that the inner sum in (1) converges with probability one for each j by the Three-Series Theorem and defines a linear process with martingale difference innovations. Thus, X_k is the sum (superposition) of linear processes, and we will call it a *superlinear process*, as in [12]. Theorem 1 provides a very large class of examples. An important special, called the *independent case*, occurs when the innovations $\xi_{i,j}$ are mutually independent and (necessarily), identically distributed for each j, and $\mathcal{F}_k$ is independent of $\sigma\{\xi_{i,j} : i > k,\ j \in J\}$ for each k. The latter condition is satisfied for $\mathcal{F}_k = \sigma\{\xi_{i,j},\ i \le k,\ j \in J\}$. Since the work of Herrndorf [7], such processes have served as a rich source of examples and counter examples for the Central Limit Theorem and Weak Invariance Principle for stationary processes. [4], [5] and [9] provide more recent ones. Our purpose here is to develop Lindeberg-Feller like conditions that are sufficient for the asymptotic normality of sums of a superlinear process and necessary for conditional asymptotic normality in the causal independent case.

Let $S_n = X_1 + \cdots + X_n$ for $n \ge 1$, and write $S_n = S_n(g)$ if the dependence on g is important. Then S_n may be written as

$$S_n = \sum_{j \in J} \sum_{i \in \mathbb{Z}} \left[\sum_{k=1}^{n} a_{k-i,j} \right] \xi_{i,j} = \sum_{j \in J} \sum_{i \in \mathbb{Z}} [b_{n-i,j} - b_{-i,j}] \xi_{i,j}, \tag{2}$$

where $b_{n,j} = a_{1,j} + \cdots + a_{n,j}$ for $n \ge 1$, $b_{0,j} = 0$, and $b_{n,j} = a_{n+1,j} + \cdots + a_{0,j}$ for $n \le -1$; then the variance of S_n,

$$\sigma_n^2 := E(S_n^2) = \sum_{j \in J} \sum_{i \in \mathbb{Z}} [b_{n-i,j} - b_{-i,j}]^2,$$

depends only on the coefficients $a_{i,j}$. Suppose throughout that

$$\lim_{n \to \infty} \sigma_n^2 = \infty, \tag{3}$$

and observe that then $\sigma_n > 0$ for all $n \ge 1$.

To state the next result, let $c_{n,j} = (b_{-n,j} + \cdots + b_{n,j})/n$, $\boldsymbol{c}_n = (c_{n,j} : j \in J)$, and regard $\boldsymbol{c}_n$ as elements of $\ell^2(J)$. Also, let $\|\cdot\|$ denote the norm in an L^2 space, $\ell^2(J)$ in Theorem 2 below and later $L^2(\Omega, \mathcal{A}, P)$. We will say that the Central Limit Theorem (CLT) holds if the distribution of S_n/σ_n converges to the standard normal distribution Φ (in symbols, $S_n/\sigma_n \Rightarrow \Phi$). Let F_j denote the marginal distribution function of the $\xi_{i,j}$.

Theorem 2 *Consider superlinear processes that satisfy (3): If*

$$\sum_{j\in J}\left\{\sum_{i=1}^{\infty}[b_{-(n+i),j}-b_{-i,j}]^2+\sum_{i=1}^{\infty}[b_{n+i,j}-b_{i,j}]^2\right\}=o(\sigma_n^2) \tag{4}$$

and the sequence $\boldsymbol{c}_n/\|\boldsymbol{c}_n\|,\ n\geq 1$, *is precompact in* $\ell^2(J)$, *then the CLT holds. Conversely, if (4) and the CLT hold for all choices of the* F_j *in the independent case, then the sequence* $\boldsymbol{c}_n/\|\boldsymbol{c}_n\|,\ n\geq 1$, *is precompact in* $\ell^2(J)$.

Observe that the conditions imposed in Theorem 2 only restrict the coefficients $a_{i,j}$ and are best possible among such conditions. In the independent case, there is a sharper result with an interesting interplay between the coefficients and the distribution functions F_j of $\xi_{0,j}, j\in J$.

Theorem 3 *For a superlinear process that has independent innovations* $\xi_{i,j}$ *and satisfies (3): If (4) holds, then the CLT holds iff*

$$L_n^*(\epsilon):=\frac{1}{\|\boldsymbol{c}_n\|^2}\sum_{j\in J}c_{n,j}^2\int_{|c_{n,j}z|>\epsilon\ \|\boldsymbol{c}_n\|\sqrt{n}}z^2F_j\{dz\}\to 0 \tag{5}$$

for each $\epsilon>0$.

A superlinear process is said to be *causal* if $a_{i,j}=0$ for all $i<0$ and $j\in J$, in which case the sequence $\ldots X_{-1},X_0,X_1,\ldots$ is adapted to the filtration $\ldots\mathcal{F}_{-1},\mathcal{F}_0,\mathcal{F}_1,\ldots$; moreover, (2) and (3) simplify since $b_{n,j}=0$ for $n<-1$. Let Φ_n denote a regular conditional distribution for S_n/σ_n given $\mathcal{F}_0$,

$$\Phi_n(\omega;z)=P\left[\frac{S_n}{\sigma_n}\leq z\right](\omega).$$

for $z\in\mathbb{R}$ and $\omega\in\Omega$. Then we will say that the Conditional Central Limit Theorem (CCLT) holds if and only if Φ_n converges weakly to Φ in probability (that is, $d(\Phi,\Phi_n)\to^{\mathrm{p}}0$ for any metric that generates the topology of weak convergence).

Theorem 4 *For a causal superlinear process that has independent innovations and satisfies (3): The CCLT holds iff (4) and (5) hold.*

Example 1 Suppose that the coefficients $a_{i,j}$ satisfy (3) and (4).

(a). If $\xi_{0,j}^2,\ j\in J$, are uniformly integrable, then (5) holds iff

$$\lim_{n\to\infty}\max_{j\in J}\frac{|c_{n,j}|}{||\mathbf{c}_n||}=0.$$

(b) if $\xi_{0,j} = \pm 2^{\frac{1}{2}j}$ with probability $1/2^{j+1}$ each and $\xi_{0,j} = 0$ otherwise, then (5) holds iff

$$\max_{j \in J} \frac{|c_{n,j}|}{||\mathbf{c}_n||} < 2^{-\frac{1}{2}j}$$

for all but a finite number of n.

(c) If $a_{i,j} = a_i b_j$ where a_i and b_j are both square summable, then X_k is a linear process, $X_k = \sum_{i \in \mathbb{Z}} a_i \zeta_{k-i}$, where $\zeta_k = \sum_{j \in J} b_j \xi_{k,j}$. So, the CLT follows from (3) and Theorem 2.5 of [8] □

Theorems 3 and 4 are established in Section 3. Section 4 contains some remarks.

2 Proofs of Theorems 1 and 2

Proof of Theorem 1. Let $\mathcal{H}_k = L^2(\Omega, \mathcal{F}_k, P)$ and $\mathcal{H}^k = \mathcal{H}_k \ominus \mathcal{H}_{k-1}$, the orthogonal complement of $\mathcal{H}_{k-1}$ in $\mathcal{H}_k$ for $k \in \mathbb{Z}$; let $e_j, j \in \mathcal{J}$, be an orthonormal basis for $\mathcal{H}^0$; and let $\xi_{i,j} = e_j \circ T^i$. Then $\xi_{i,j},\ j \in \mathcal{J}$, is an orthonormal basis of $\mathcal{H}_i$ for each i, and $\xi_{i,j},\ i \in \mathbb{Z}$, are martingale differences with respect to $\mathcal{F}_i$ for each j, since $E(\xi_{i,j}|\mathcal{F}_{i-1}) = E(e_j \circ T^i|\mathcal{F}_{i-1}) = E(e_j|\mathcal{F}_{-1}) \circ T^i = 0$ for each i and j. The random variables $\xi_{i,j}$ are pairwise orthogonal. For if $i \neq i'$, then $\xi_{i,j}$ and $\xi_{i',j'}$ are in mutually orthogonal subspaces; and if $i = i'$ but $j \neq j'$, then $\xi_{i,j}$ and $\xi_{i,j'}$ are orthogonal by construction.

Next let Π_i denote the projection onto $\mathcal{H}^i$, given by

$$\Pi_i Y = E(Y|\mathcal{F}_i) - E(Y|\mathcal{F}_{i-1})$$

for $Y \in L^2(\Omega, \mathcal{A}, P)$. Then the conditions that g be $\mathcal{F}_\infty$ measurable and $E(g|\mathcal{F}_{-\infty}) = 0$ imply

$$g = \sum_{i \in \mathbb{Z}} \Pi_i g.$$

Since $\xi_{i,j},\ j \in \mathcal{J}$, is an orthonormal basis for $\mathcal{H}^i$ for each $i \in \mathbb{Z}$, there are square summable coefficients $a_{i,j},\ j \in \mathcal{J}$, for which

$$\Pi_{-i} g = \sum_{j \in \mathcal{J}} a_{i,j} e_j \circ T^{-i}. \tag{6}$$

The set $\mathcal{J}$ may be uncountable, but at most countably many of the $a_{i,j}$ can be nonzero in (6). Letting $J_i = \{j : a_{i,j} \neq 0\}$ and $J = \cup_{i \in \mathbb{Z}} J_i$, it follows that J is countable and

$$g = \sum_{i \in \mathbb{Z}} \sum_{j \in J} a_{i,j} \xi_{-i,j}.$$

This is Equation (1) for $k = 0$ from which the general case follows easily. Finally, the array $a_{i,j}$ is square summable by the orthogonality of $\xi_{i,j}$, and $\|g\|^2 = \sum_{i \in \mathbb{Z}} \sum_{j \in J} a_{i,j}^2$. □

Some Algebra. It follows directly from (2) that

$$E(S_n|\mathcal{F}_m) = \sum_{j\in J}\sum_{i\le m}[b_{n-i,j} - b_{-i,j}]\xi_{i,j}$$

for $n \ge 0$ and $m \in \mathbb{Z}$. Thus the sums that appear on the left side of (4) are

$$\|E(S_n|\mathcal{F}_0)\|^2 = \sum_{j\in J}\sum_{i\le 0}[b_{n-i,j} - b_{-i,j}]^2 = \sum_{j\in J}\sum_{i=0}^{\infty}[b_{i+n,j} - b_{i,j}]^2$$

and

$$\|S_n - E(S_n|\mathcal{F}_n)\|^2 = \sum_{j\in J}\sum_{i=n+1}^{\infty}[b_{n-i,j} - b_{-i,j}]^2 = \sum_{j\in J}\sum_{i=1}^{\infty}[b_{-(i+n),j} - b_{-i,j}]^2.$$

Recall the notation $c_{n,j} = (b_{-n,j} + \cdots + b_{n,j})/n$, and let

$$D_{n,k} = \sum_{j\in J} c_{n,j}\xi_{k,j} \quad \text{and} \quad M_{n,k} = D_{n,1} + \cdots + D_{n,k}. \tag{7}$$

Then $D_{n,k},\ k \in \mathbb{Z}$ form a stationary sequence of martingale differences with respect to $\mathcal{F}_k,\ k \in \mathbb{Z}$, for each n. If (3) and (4) hold, then there is a slowly varying function ℓ for which $\sigma_n^2 = n\ell(n)$ and

$$\max_{k\le n}\|S_k - M_{n,k}\| = o(\sigma_n) \tag{8}$$

by Theorem 1 of [11] in the causal case and Theorem 3 of [10] otherwise. So, if (3) and (4) hold, then $S_n/\sigma_n \Rightarrow \Phi$ if and only if $M_{n,n}/\sigma_n \Rightarrow \Phi$, and the CLT holds if the D_{nk} satisfy the conditions of the Martingale Central Limit Theorem (for example, [2], pp. 476-478),

$$\lim_{n\to\infty}\frac{1}{\sigma_n^2}\sum_{k=1}^{n}E(D_{nk}^2|\mathcal{F}_{k-1}) = 1, \tag{9}$$

and

$$L_n(\epsilon) := \frac{1}{\sigma_n^2}\sum_{k=1}^{n}E\left[D_{nk}^2\mathbf{1}_{\{|D_{nk}|>\epsilon\sigma_n\}}|\mathcal{F}_{k-1}\right] \to 0 \tag{10}$$

in probability as $n \to \infty$ for each $\epsilon > 0$. Observe that if (4) holds, then

$$\sigma_n^2 = E(S_n^2) = E(M_{nn}^2) + o(\sigma_n^2) = n\sum_{j\in J}c_{n,j}^2 + o(\sigma_n^2) = n\|\boldsymbol{c}_n\|^2 + o(\sigma_n^2) \tag{11}$$

by (8).

Proof of Theorem 2. If (4) holds, then so do (8) and (11), so that $\|\boldsymbol{c}_n\| \ne 0$ for all large n, say $n \ge n_0$, and

$$\frac{S_n}{\sigma_n} - \frac{M_{n,n}}{\sqrt{n}\|\boldsymbol{c}_n\|} \to 0$$

in probability.

It is first shown that $M_{n,n}/(\sqrt{n}\|\boldsymbol{c}_n\|) \Rightarrow \Phi$ if $\boldsymbol{c}_n/\|\boldsymbol{c}_n\|,\ n \geq n_0$, is precompact in $\ell^2(J)$. Define $\Psi : \ell^2(J) \to \mathcal{H}^0$ by $\Psi(\boldsymbol{a}) = \sum_{j\in J} a_j \xi_{0,j}$ for $\boldsymbol{a} \in \ell^2(J)$. Then Ψ is an isomorphism and, in particular, maps the surface of the unit ball in $\ell^2(J)$, $\{\boldsymbol{a} \in \ell^2(J) : \|\boldsymbol{a}\| = 1\}$ onto the surface of the unit ball in $\mathcal{H}^0$. Let $\mathcal{C}$ be the closure of $\{\boldsymbol{c}_n/\|\boldsymbol{c}_n\| : n \geq n_0\}$. Then $\mathcal{C}$ is compact by assumption and, therefore, so is $\mathcal{K} = \Psi(\mathcal{C})$. So, given any $\epsilon > 0$, there is a finite collection $\{f_1, \ldots, f_m\} \subseteq \mathcal{H}^0$ for which $\|f_i\| = 1$ for all i, and $\min_{1\leq i\leq m} \|h - f_i\| \leq \epsilon$ for all $h \in \mathcal{K}$. Then, recalling the notation $S_n(f) = f \circ T + \cdots + f \circ T^n$, $S_n(f_i)/\sqrt{n} \Rightarrow \Phi$ as $n \to \infty$ for all $i = 1, \ldots, m$, by the Martingale Central Limit Theorem applied to the stationary sequence of martingale differences $f_i \circ T^k$. Observe that $D_{n,0}/\|\boldsymbol{c}_n\| \in \mathcal{K}$ for $n \geq n_0$. So, given $\epsilon > 0$, there are i_n for which $1 \leq i_n \leq m$ and $\|D_{n,0}/\|\boldsymbol{c}_n\| - f_{i_n}\| \leq \epsilon$ for all $n \geq n_0$, Write

$$\frac{1}{\sqrt{n}\|\boldsymbol{c}_n\|} M_{n,n} = \frac{1}{\sqrt{n}} S_n(f_{i_n}) + \frac{1}{\sqrt{n}} S_n\left(\frac{D_{n,0}}{\|\boldsymbol{c}_n\|} - f_{m_n}\right).$$

The distribution of the first term on the right converges to Φ, and the expected square of the second term is at most ϵ^2. The asymptotic normality of the term on the left and, therefore, S_n/σ_n, follows.

The proof of the converse is similar to that of Theorem 3 of [12]. Let $\boldsymbol{d}_n = \boldsymbol{c}_n/\|\boldsymbol{c}_n\|$ for $n \geq n_0$. Then $\boldsymbol{d}_n,\ n \geq 1$, are weakly precompact, and it suffices to show that any weak limit point is a strong limit point. Let $\boldsymbol{d} \in \ell^2(J)$ be an arbitrary weak limit point and let $\mathbb{N}_0$ be a subsequence for which $\text{weaklim}_{n\in\mathbb{N}_0} \boldsymbol{d}_n = \boldsymbol{d}$. Then $\lim_{n\in\mathbb{N}_0} d_{n,j} = d_j$ for all j and, therefore,

$$\lim_{n\in\mathbb{N}_0} \sum_{j=1}^{j_n} [d_{n,j} - d_j]^2 = 0$$

for some subsequence $j_n \to \infty$. By thinning the subsequence $\mathbb{N}_0$, if necessary, we may suppose that $j_n, n \in \mathbb{N}_0$, are strictly increasing. There is a strictly decreasing sequence $1 > q_1 > q_2 > \cdots$ for which $\lim_{n\in\mathbb{N}_0} nq_{j_n} = 0$. Let $p_j = q_j - q_{j+1}$; let F_j be the distribution which assigns mass $p_j/2$ to $\pm 1/\sqrt{p_j}$ and mass $1 - p_j$ to 0; consider independent innovations $\xi_{i,j} \sim F_j$; and let $\zeta_{n,j} = \xi_{1,j} + \cdots + \xi_{n,j}$ and $\tilde{M}_{n,n} = \sum_{j=1}^{j_n} d_{n,j}\zeta_{n,j}$. Then $P[\zeta_{n,j} \neq 0] \leq np_j$, and

$$P\left[\frac{M_{n,n}}{\|\boldsymbol{c}_n\|} \neq \tilde{M}_{n,n}\right] \leq nq_{j_n} \to 0$$

as $n \to \infty$. So, if $S_n/\sigma_n \Rightarrow \Phi$, then also $\tilde{M}_{n,n}/\sqrt{n} \Rightarrow \Phi$ and, therefore,

$$\liminf_{n\in\mathbb{N}_0} \sum_{j=1}^{j_n} d_{n,j}^2 = \liminf_{n\in\mathbb{N}_0} \frac{1}{n} E(\tilde{M}_{n,n}^2) \geq 1.$$

It follows easily that $\lim_{n\in\mathbb{N}_0} \sum_{j=j_n+1}^{\infty} d_{n,j}^2 = 0$ and, therefore, that $\lim_{n\in\mathbb{N}_0} \boldsymbol{d}_n = \boldsymbol{d}$ in $\ell^2(J)$. Weak compactness then follows, since $\boldsymbol{d}$ was an arbitrary weak limit point. $\square$

3 Proofs of Theorems 3 and 4

Some Inequalities. The following version of the Baum-Katz inequalities, [1], is needed: Let $Z_j, j \in J$, *be independent random variables with means* 0 *and variances* $b_j^2, j \in J$ *for which* $\sum_{j\in J} b_j^2 < \infty$*; and let* $Y = \sum_{j\in J} Z_j$ *and* $\mathbf{b} = (b_j : j \in J)$*. Then*

$$P\left[|Y| > 3x\right] \leq \left(\frac{\|\mathbf{b}\|^2}{x^2}\right)^2 + \sum_{j\in J} P\left[|Z_j| > x\right] \tag{12}$$

for all $x > 0$. To prove (12) suppose first that J is a finite set, say $J = \{1, \ldots, n\}$, and let $Y_k = Z_1 + \cdots + Z_k$ for $k \leq n$. For this case, we prove a stronger version in which $|Y|$ is replaced by $\max_{k\leq n} |Y_k|$ on the left side of (12). Let B be the event that $\max_{k\leq n} |Z_k| \leq x$ and let $\tau = \inf\{k : |Y_k| > x\}$. If B occurs and $\max_{k\leq n} |Y_k| > 3x$, then clearly $\tau < n$, $Y_\tau < 2x$, and $\max_{\tau<k\leq n} |Y_k - Y_\tau| > x$. Letting $\mathcal{G}_k = \sigma\{Z_1, \ldots, Z_k\}$,

$$P\left[\max_{\tau<k\leq n} |Y_k - Y_\tau| > x \mid \mathcal{G}_\tau\right] \leq \frac{b_{\tau+1}^2 + \cdots + b_n^2}{x^2} \leq \frac{\|\mathbf{b}\|^2}{x^2},$$

by Kolmogorov's inequality and, therefore,

$$\begin{aligned} P\left[\tau < n,\ \max_{\tau<k\leq n} |Y_k - Y_\tau| > x\right] &\leq \int_{\tau<n} P\left[\max_{\tau<k\leq n} |Y_k - Y_\tau| > x \mid \mathcal{G}_\tau\right] dP \\ &\leq \left(\frac{\|\mathbf{b}\|^2}{x^2}\right) P[\tau < n] \ \leq \left(\frac{\|\mathbf{b}\|^2}{x^2}\right)^2, \end{aligned}$$

where the last inequality follows by invoking Kolmogorov's inequality again. The inequality (12) for finite J then follows from

$$P\left[\max_{k\leq n} |Y_k| > 3x\right] \leq P\left[\tau < n,\ \max_{\tau<k\leq n} |Y_k - Y_\tau| > x\right] + \sum_{k=1}^{n} P[|Z_k| > x].$$

The case of countably infinite J then follows easily by applying (12) to finite subsets I that increase to J. Letting $Z_j = c_{n,j}\xi_{k,j}$ in (12) then yields

$$P[|D_{n,k}| > 3x] \leq \left(\frac{\|\boldsymbol{c}_n\|^2}{x^2}\right)^2 + \sum_{j\in J} P[|c_{n,j}\xi_{k,j}| > x]. \tag{13}$$

A second inequality relates $\int_{|Y|>c} Y^2 dP$ and $\int_{|Z_j|>c} Z_j^2 dP$. The following conditional probability is needed, in which $Y_{-j} = \sum_{i\neq j} Z_i$: If $Z_j \geq c > 0$, then

$$P\left[Y \geq \frac{1}{2}Z_j \mid Z_j\right] = P\left[Y_{-j} \geq -\frac{1}{2}Z_j \mid Z_j\right] \geq \frac{Z_j^2}{Z_j^2 + 4\|\mathbf{b}\|^2} \geq \frac{c^2}{c^2 + 4\|\mathbf{b}\|^2}$$

by the one-sided version of Chebyshev's inequality. So,

$$\int_{Z_j>c} Z_j^2 dP \le \left(1+\frac{4\|\mathbf{b}\|^2}{c^2}\right)\int_{c<Z_j<2Y} Z_j^2 dP$$
$$\le \left(1+\frac{4\|\mathbf{b}\|^2}{c^2}\right)\int_{Y>\frac{1}{2}c} 2YZ_j dP.$$

and

$$\sum_{j\in J}\int_{Z_j>c} Z_j^2 dP \le 2\left(1+\frac{4\|\mathbf{b}\|^2}{c^2}\right)\int_{Y>\frac{1}{2}c} Y^2 dP.$$

Combining this with its dual, obtained by replacing Z_j with $-Z_j$,

$$\sum_{j\in J}\int_{|Z_j|>c} Z_j^2 dP \le 2\left(1+\frac{4\|\mathbf{b}\|^2}{c^2}\right)\int_{|Y|>\frac{1}{2}c} Y^2 dP. \tag{14}$$

Proof of Theorem 3. In the independent case the conditional expectations in (9) and (10) reduce to unconditional expectations, so that (9) and (10) are necessary and sufficient for $M_{n,n}/\sigma_n \Rightarrow \Phi$, by the Lindeberg-Feller Theorem [6], pp. 101-102. So, if (3) and (4) hold, then $S_n/\sigma_n \Rightarrow \Phi$ iff (9) and (10) hold. Thus, it suffices to show that (4) and (5) imply (9) and (10) and then that (4) and (10) imply (5). In the independent case, (9) is easily verified for the process (1), since

$$E(D_{n,k}^2|\mathcal{F}_{k-1}) = E(D_{n,k}^2) = \sum_{j\in J} c_{n,j}^2 = \|\boldsymbol{c}_n\|^2,$$

and the sum in (9) becomes $n\|\boldsymbol{c}_n\|^2/\sigma_n^2$, which converges to 1 if (4) holds. Condition (10) presents more of a challenge.

Using the independence and stationarity, and then integrating by parts, one has

$$L_n(\epsilon) = \frac{n}{\sigma_n^2}E\left[D_{n,1}^2\mathbf{1}_{\{|D_{n,1}|>\epsilon\sigma_n\}}\right]$$
$$= n\epsilon^2 P\left[|D_{n,1}|>\epsilon\sigma_n\right] + \frac{n}{\sigma_n^2}\int_{\epsilon\sigma_n}^{\infty} 2xP\left[|D_{n,1}|>x\right]dx$$
$$= I_n + II_n,$$

say. Let F_j denote the distribution function of ξ_{ij}. Then, using (11), (12) and (13),

$$\begin{aligned} I_n &\le n\epsilon^2\left(\frac{9\|\boldsymbol{c}_n\|^2}{\epsilon^2\sigma_n^2}\right)^2 + n\epsilon^2\sum_{j\in J}P\left[|c_{n,j}\xi_{1,j}|>\frac{1}{3}\epsilon\sigma_n\right] \\ &\le 81n\left(\frac{\|\boldsymbol{c}_n\|^4}{\epsilon^2\sigma_n^4}\right) + \frac{9n}{\sigma_n^2}\sum_{j\in J}c_{n,j}^2\int_{|c_{n,j}z|>\frac{1}{3}\epsilon\sigma_n} z^2F_j\{dz\} \\ &\le 81n\left(\frac{\|\boldsymbol{c}_n\|^4}{\epsilon^2\sigma_n^4}\right) + \frac{9n\|\boldsymbol{c}_n\|^2}{\sigma_n^2}L_n^*(\epsilon/4), \end{aligned} \tag{15}$$

for all sufficiently large n, where L_n^* is as in (5), and the first term in (15) approaches zero as $n \to \infty$, since $\sigma_n^2 \sim n\|\boldsymbol{c}_n\|^2$ by (11). Similarly, using (13) again and another integration by parts yield

$$\begin{aligned}
II_n &= \frac{9n}{\sigma_n^2}\int_{\frac{1}{3}\epsilon\sigma_n}^{\infty} 2xP[|D_{n1}| > 3x]dx \\
&\le \frac{9n}{\sigma_n^2}\int_{\frac{1}{3}\epsilon\sigma_n}^{\infty} 2x\left(\frac{\|\boldsymbol{c}_n\|^2}{x^2}\right)^2 dx + \frac{9n}{\sigma_n^2}\sum_{j\in J}\int_{\frac{1}{3}\epsilon\sigma_n}^{\infty} 2xP[|c_{n,j}\xi_{1,j}| > x]dx \\
&\le \frac{81n}{\sigma_n^2}\left(\frac{\|\boldsymbol{c}_n\|^4}{\epsilon^2\sigma_n^2}\right) + \frac{9n}{\sigma_n^2}\sum_{j\in J} c_{n,j}^2 \int_{|c_{n,j}z|>\frac{1}{3}\epsilon\sigma_n} z^2 F_j\{dz\} \\
&\le 81n\left(\frac{\|\boldsymbol{c}_n\|^4}{\epsilon^2\sigma_n^4}\right) + \frac{9n\|\boldsymbol{c}_n\|^2}{\sigma_n^2}L_n^*(\epsilon/4)
\end{aligned}$$

for all sufficiently large n. Again, the first term in the last line approaches zero by (11). So, if (4) and (5) hold, then so do (9) and (10). To see that (4) and (10) imply (5), let $Z_j = c_{n,j}\xi_{1,j}$ in (14). Then, by (14),

$$\begin{aligned}
L_n^*(\epsilon) &= \frac{1}{\|\boldsymbol{c}_n\|^2}\sum_{j\in J} E\left[Z_j^2 \mathbf{1}_{\{|Z_j|\ge\epsilon\|\boldsymbol{c}_n\|\sqrt{n}\}}\right] \\
&\le \frac{2}{\|\boldsymbol{c}_n\|^2}\left(1+\frac{4}{n\epsilon^2}\right) E\left[D_{n1}^2 \mathbf{1}_{\{|D_{n1}|\ge\frac{1}{2}\epsilon\|\boldsymbol{c}_n\|\sqrt{n}\}}\right] \le 4L_n(\epsilon/4)
\end{aligned}$$

for all large n. So, if the CCLT holds, then $L_n(\epsilon) \to 0$ for all $\epsilon > 0$ and, hence $L_n^*(\epsilon) \to 0$ for all $\epsilon > 0$. □

Proof of Theorem 4. In the causal case, (4) is a necessary condition for the CCLT, and relations (8), (9), and (10) are necessary and sufficient for the CCLT by Lemma 2 and Theorem 2 of [11]. Theorem 4 now follows easily. For if (4) and (5) hold, then so do (8), (9), and (10), as has just been shown, establishing the sufficiency of (4) and (5). Conversely, if the CCLT holds, then so do (4), (8), (9), and (10), by Lemma 2 and Theorem 2 of [11], and this implies (5) as has just been shown. □

4 Remarks

The weak invariance principle, the convergence of $S_{\lfloor nt\rfloor}/\sigma_n$ to Brownian motion, does not follow from the conditions imposed in Theorem 3. An example of superlinear process for which $\sigma_n^2 \sim n$, (4) and the CLT hold, but the weak invariance principle does not may be found in the proof of Proposition 4 of [4]. An example of a causal linear process for which $\sigma_n = o(\sqrt{n})$, the CCLT holds, but the weak invariance principle fails may be found in [11].

A linear process, $X_k = \sum_{i\in\mathbb{Z}} a_i\xi_{k-i}$, where $\ldots a_{-1}, a_0, a_1, \ldots$ are square summable and $\ldots \xi_{-1}, \xi_0, \xi_1, \ldots$ are a stationary sequence of martingale differences with $E(\xi_i^2) = 1$, is also

a superlinear process in which J is a singleton. For a causal linear process that satisfies (3) and has independent innovations, Wu and Woodroofe [11] showed that the CCLT holds iff

$$\|E(S_n|\mathcal{F}_0)\| = o(\sigma_n) \tag{16}$$

which is just (4) in the causal case. This is a simple corollary to Theorem 4, since (5) is trivially satisfied when J is a singleton. That (16) is not sufficient for causal superlinear processes is clear from Theorems 2 and 4. A specific example is provided by Klicnarova and Volný [9]. They construct an example in which

$$\lim_{n\to\infty} \frac{\sigma_n^2}{n} = 1, \qquad \|E(S_n|\mathcal{F}_0)\| = O\left[\sqrt{\frac{n}{\log(n)}}\right],$$

and the distribution of $S_n/\sqrt{n}$ has two distinct limit points, one normal and the other a symmetrized Poisson.

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