**Probability Models**

**Important Concepts**

Read Chapter 2

- Probability Models
- Examples
  - The Classical Model
  - Discrete Spaces
- Elementary Consequences of the Axioms
- The Inclusion Exclusion Formulas
- Some Indiscrete Models
- Monotone Sequences and Continuity

**Experiments**

**Phenomena**

- Unpredictable in detail
- The set of possible outcomes in known.

**Examples**

a) Scientific experiments
b) Games of chance
c) Human performance
d) Financial indices
e) The Weather

**Events and The Sample Space**

**The Sample Space.** Let \( \Omega \) denote the set of possible outcomes for a given experiment.

**Events:** Subsets of the sample space, \( A, B, C \subseteq \Omega \).

**Example:** Coin Tossing. \( \Omega = \{hH, hT, tH, tT\} \) and \( A = \{hT, tH\} \).

**The Algebra of Events** Set theory operations on events—for example,

\[
A \cup B = \{\omega : \omega \in A \text{ or } \omega \in B\},
\]

\[
AB = \{\omega : \omega \in A \text{ and } \omega \in B\},
\]

\[
A^c = \{\omega : \omega \notin A\},
\]

\[
B - A = BA^c
\]

**The Model**

**Three Elements**

- The sample space: \( \Omega \neq \emptyset \).
- Events: Subsets of \( A, B, C, \ldots \subseteq \Omega \).
- Probability: Let \( \mathcal{A} \) be the class of events, and let \( P : \mathcal{A} \to \mathbb{R} \) must satisfy

\[
P(\Omega) = 1, \quad (1)
\]

\[
0 \leq P(A) \leq 1, \quad (2)
\]

\[
P(A \cup B) = P(A) + P(B) \quad (3)
\]

whenever \( A \) and \( B \) are events for which \( AB = \emptyset \).

**Notes**

a) Probability is a property of events.

b). (1), (2), and (3) are axioms and admit various interpretations.
The Classical Model
Games of Chance

The Model. \( \Omega \) is a finite set; \( A \) is the class of all subsets of \( \Omega \); and
\[
P(A) = \frac{\#A}{\#\Omega}.
\]

Example: Roulette
\[\Omega = \{0, 00, 1, 2, 3, 4, \ldots, 35, 36\}\]
and
\[
P(\{\text{Red Outcome}\}) = \frac{18}{38} = \frac{9}{19}.
\]

The Birthday Problem

Q: If \( n \) people gather, what is the probability that no two have the same birthday?

A: Regard the birthdays of the \( n \) people as a sample w.r. from \( \{1, 2, \ldots, 365\} \) (ignoring leap year). Then \( \Omega \) is all lists \( \omega = (i_1, \ldots, i_n) \)
and \( \#\Omega = 365^n \). Let
\[A = \{\omega : i_j \neq i_k \text{ all } j \neq k\}.
\]
Then \( A \) consists of all permutations of \( n \) days, \( \#A = (365)_n \), and
\[
P(A) = \frac{(365)_n}{365^n} = p_n \text{ say.}
\]

Some Values

<table>
<thead>
<tr>
<th>( n )</th>
<th>8</th>
<th>16</th>
<th>24</th>
<th>32</th>
<th>40</th>
</tr>
</thead>
<tbody>
<tr>
<td>( p_n )</td>
<td>.924</td>
<td>.716</td>
<td>.462</td>
<td>.247</td>
<td>.109</td>
</tr>
</tbody>
</table>

Discrete Probability Models

Suppose \( \Omega = \{\omega_1, \omega_2 \cdots\} \), finite or infinite; let
\[p : \Omega \to \mathbb{R},
\]
satisfy
\[
p(\omega) \geq 0 \text{ for all } \omega,
\]
\[
\sum_{\omega \in \Omega} p(\omega) = 1.
\]
Let
\[
P(E) = \sum_{\omega \in E} p(\omega)
\]
for \( E \subseteq \Omega \).

Notes a) Then (1), (2), and (3) hold.
b) \( p(\omega) = P(\{\omega\}) \).

Example. In the classical model, \( p(\omega) = 1/\#\Omega \).

On Infinite Sums

If \( x_1, x_2, \cdots \in \mathbb{R} \), then
\[
\sum_{k=1}^{\infty} x_k = \lim_{n \to \infty} \sum_{k=1}^{n} x_k,
\]
provided that the limit exists.

Examples a). If \(-1 < x < 1\), then
\[
\sum_{k=1}^{\infty} x^{k-1} = \frac{1}{1-x}.
\]
b). For any \( x \),
\[
\sum_{k=0}^{\infty} \frac{1}{k!} x^k = e^x.
\]

Alternative Notation: If \( A = \{x_1, x_2, \cdots\} \), and \( f : A \to [0, \infty) \), write
\[
\sum_{x \in A} f(x) = \sum_{k=1}^{\infty} f(x_k).
\]
Waiting for Success
Play Roulette Until a You Win
Betting on Red

Let

\[ r = \frac{9}{19}, \]
\[ q = 1 - r = \frac{10}{19}, \]

and

\[ \Omega = \{1, 2, \cdots \} \]

Then, intuitively,

\[ p(1) = r, \]
\[ p(2) = qr, \]
\[ p(3) = q^2r, \]
\[ \cdots, \]
\[ p(\omega) = rq^{\omega-1}. \]

Then

\[ \sum_{\omega \in \Omega} p(\omega) = \sum_{\omega = 1}^{\infty} rq^{\omega-1} \]
\[ = \frac{r}{1 - q} \]
\[ = 1. \]

Let

\[ P(A) = \sum_{\omega \in A} p(\omega). \]

Amusing Calculation: Let Odd = \{1, 3, \cdots \}.

Then

\[ P(\text{Odd}) = \sum_{k=0}^{\infty} rq^{(2k+1)-1} \]
\[ = r \sum_{k=0}^{\infty} q^{2k} \]
\[ = \frac{r}{1 - q^2} \]
\[ = \frac{19}{29}. \]

The Objective Interpretation

Thought Experiment: Imagine the experiment repeated \( N \) times. For an event \( A \), let

\[ N_A = \# \text{ occurrences of } A. \]

Then

\[ P(A) = \lim_{N \to \infty} \frac{N_A}{N}. \]

Example: Coin Tossing

<table>
<thead>
<tr>
<th>( N )</th>
<th>( N_H / N )</th>
</tr>
</thead>
<tbody>
<tr>
<td>100</td>
<td>.550</td>
</tr>
<tr>
<td>1000</td>
<td>.493</td>
</tr>
<tr>
<td>10000</td>
<td>.514</td>
</tr>
<tr>
<td>100000</td>
<td>.503</td>
</tr>
</tbody>
</table>

Note: Consistent with \( P(H) = .5 \).

Example. In many roulette games, about 9/19 will result in red.
Consequences of the Axioms

Suppose that $P$ satisfies (1), (2), and (3).

If $A$ and $B$ are events for which $A \subseteq B$, then

$$P(B - A) = P(B) - P(A).$$  \hfill (4)

For any event $A$,

$$P(A^c) = 1 - P(A).$$  \hfill (5)

In particular,

$$P(\emptyset) = 0.$$  \hfill (6)

For any events $A$ and $B$,

$$P(A \cup B) = P(A) + P(B) - P(AB)$$  \hfill (7)

If $A_1, \ldots, A_m$ are any $m$ events, then

$$P\left(\bigcup_{i=1}^{m} A_i\right) \leq \sum_{i=1}^{m} P(A_i),$$  \hfill (8)

with equality if $A_iA_j = \emptyset$ whenever $i \neq j$.

Proofs

If $A \subseteq B$, then

$$B = A \cup (B - A)$$

and $A \cap (B - A) = \emptyset$. So,

$$P(B) = P(A) + P(B - A),$$

by (3) and, therefore,

$$P(B - A) = P(B) - P(A).$$  \hfill (4)

For (5), $A^c = \Omega - A$. So,

$$P(A^c) = P(\Omega) - P(A) = 1 - P(A).$$  \hfill (5)

For (6), $P(\emptyset) = P(\Omega^c) = 0$.

Example. In the birthday problem, the probability that at least two people have the same birthday is $A^c$, and

$$P(A^c) = 1 - P(A) = 1 - \frac{(365)^n}{365^n}.$$  

More on Unions

If $A_1, \ldots, A_m$ are events, let

$$\sigma_1 = \sum_{i=1}^{m} P(A_i),$$

$$\sigma_2 = \sum_{1 \leq i < j \leq m} P(A_iA_j),$$

$$\sigma_3 = \sum_{1 \leq i < j < k \leq m} P(A_iA_jA_k),$$

$$\sigma_k = \sum_{1 \leq i_1, \ldots, i_k \leq m} P(A_{i_1}\cdots A_{i_k}),$$

$$\sigma_m = P(A_1A_2\cdots A_m).$$

Then

$$P\left(\bigcup_{i=1}^{m} A_i\right) = \sigma_1 - \sigma_2 + \cdots \pm \sigma_m.$$  

The Matching Problem

Let $\Omega$ be all permutations

$$\omega = (i_1, \cdots, i_n)$$

of $1, 2, \cdots, n$. Thus,

$$\Omega = n!.$$ 

Let

$$A_j = \{\omega: i_j = j\} \quad A = \bigcup_{i=1}^n A_i.$$ 

Then

$$\sigma_k = \binom{n}{k} P(A_1 \cdots A_k),$$

by symmetry.

Examples. Gift exchange

Here

$$P(A_1) = \frac{1 \times (n-1)!}{n!} = \frac{1}{n},$$

$$P(A_1 A_2) = \frac{(n-2)!}{n!} = \frac{1}{(n)_2},$$

$$\quad \cdots,$$

$$P(A_1 \cdots A_k) = \frac{(n-k)!}{n!} = \frac{1}{(n)_k},$$

for $k = 1, \cdots, n$. So,

$$\sigma_k = \binom{n}{k} (n)_k = \frac{1}{k!},$$

$$P(A) = \sigma_1 - \sigma_2 + \cdots \pm \sigma_n$$

$$= \sum_{k=1}^n \frac{1}{k!} (-1)^{k-1},$$

and

$$P(A) = 1 - \sum_{k=0}^n \frac{1}{k!} (-1)^k \approx 1 - \frac{1}{e}.$$ 

Note: Accurate to three places if $n \geq 6.$

Refinements

More on Events. Not all subsets of $\Omega$ need be events; but the class of events must be closed under union, intersection, and complementation.

More on the Third Axiom. A stronger version of (3) requires

$$P(\bigcup_{k=1}^\infty A_k) = \sum_{k=1}^\infty P(A_k), \quad (3*)$$

whenever $A_1, A_2, \cdots$ are mutually exclusive events (that is, $A_i A_j = \emptyset$ for $i \neq j$).

Remark: $(3*)$ implies (3).

Proposition. The discrete probability models satisfy $(3*)$, as well as (3).

Proof. Omitted

Some Indiscrete Models

Intervals

$$(a, b) = \{x: a < x < b\},$$

$$(a, b] = \{x: a < x \leq b\},$$

$$(a, b] = \{x: a \leq x < b\},$$

$$(a, b] = \{x: a < x \leq b\},$$

Densities. Let $\Omega$ be an interval and $f$ a function for which $f(\omega) \geq 0$ and

$$\int_{\Omega} f(\omega) d\omega = 1.$$ 

Then let

$$P(I) = \int_I f(\omega) d\omega$$

for intervals $I$ and extend $f$ to a larger class of events using the axioms.

Example The Uniform Spinner. Let $\Omega = (-\pi, \pi]$ and $f(\omega) = 1/2\pi$. Then

$$P((a, b)) = \cdots = P([a, b]) = \frac{b - a}{2\pi}.$$
Amusing Calculation
About the Extension Process

Note. For any $\omega$,
\[ P(\{\omega\}) = P(\omega,\omega) = \int_{\omega} f(\omega')d\omega' = 0. \]

If
\[ C = \{\omega_1, \omega_2, \cdots\}, \]
then
\[ P(C) = \sum_{i=1}^{\infty} P(\{\omega_i\}) = 0. \]

The probability of a rational outcome is zero.

Monotone Sequences

Events $A_1, A_2, \cdots$ are **increasing** if
\[ A_1 \subseteq A_2 \subseteq \cdots \]
and **decreasing** if
\[ A_1 \supseteq A_2 \supseteq \cdots. \]

The limit of an increasing (respectively, decreasing) sequence is
\[ A_\infty = \bigcup_{k=1}^{\infty} A_k, \]
respectively,
\[ A_\infty = \bigcap_{k=1}^{\infty} A_k. \]

Example. If $\Omega = \mathbb{R}$ and
\[ A_k = (-\infty, \frac{1}{k}) = \{\omega : \omega < \frac{1}{k}\}, \]
then $A_k$ are decreasing and
\[ A_\infty = \{\omega : \omega < \frac{1}{k} \text{ for all } k\} = (-\infty, 0]. \]

De Morgan’s Laws. For any events $A_i, i = 1, \cdots, n,$
\[ (\bigcup_{i=1}^{n} A_i)^c = \bigcap_{i=1}^{n} A_i^c, \]
\[ (\bigcap_{i=1}^{n} A_i)^c = \bigcup_{i=1}^{n} A_i^c. \]

Also true if $n = \infty$. Proof-e.g., $\omega \in (\bigcup_{i=1}^{n} A_i)^c$ iff $\omega \notin \bigcup_{i=1}^{n} A_i$ iff $\omega \notin A_i$ for any $i$ iff $\omega \in \bigcup_{i=1}^{n} A_i^c$.

Corollary. If $A_1, A_2, \cdots$ is increasing or decreasing, then then
\[ (A_\infty)^c = (A^c)_\infty. \]

The Monotone Sequences Theorem

Suppose that $P$ satisfies (1), (2), and (3$^0$). Then
$P$ satisfies (3) iff
\[ P(A_\infty) = \lim_{n \to \infty} P(A_n), \]
whenever $A_1, A_2, \cdots$ is an increasing, or decreasing, sequence of events.

Proof. Later, or see the text.

Remarks a). Type of continuity.

b). Equivalent to (3).

c). Useful.