Combinatorics
Counting
An Overview

- Introductory Example
- What to Count
  Lists
  Permutations
  Combinations.
- The Basic Principle
- Counting Formulas
- The Binomial Theorem.
- Partitions
- Solutions

Example
As I was going to St. Ives
I met a man with seven wives
Every wife had seven sacks
Every sack had seven cats
Every cat had seven kits
Kits, cats, sacks, wives
How many were going to St. Ives?

Lists
Can be Counted Easily

Order Pairs: \((x, y) = (w, z)\) iff \(w = x\) and \(z = y\).

Ordered Triples: \((x, y, z) = (u, v, w)\) iff \(u = x, v = y,\) and \(w = z\).

Lists of Length \(r\) (AKA Order \(r\)-tuples):
\((x_1, \ldots, x_r) = (y_1, \ldots, y_s)\)
iff \(s = r\) and \(y_i = x_i\) for \(i = 1, \ldots, r\).

Example: License Plates. A license plate has the form \(LMNxyz\), where
\(L, M, N \in \{A, B, \ldots, Z\}\),
\(x, y, z \in \{0, 1, \ldots, 9\}\)
and, so, is a list of length six.

Ans: None.
How many were going the other ways?
7 Wives.
7 \times 7 = 49 sacks.
49 \times 7 = 343 cats.
343 \times 7 = 2401 kits.
Total = 2800.
Basic Principle of Combinatorics
The Multiplication Principle

For Two: If there are \( m \) choices for \( x \) and then \( n \) choices for \( y \), then there are \( m \times n \) choices for \((x, y)\).

For Several: If there are \( n_i \) choices for \( x_i \), \( i = 1, \ldots, r \), then there are
\[
\prod_{i=1}^{r} n_i
\]
choices for \((x_1, \ldots, x_r)\).

Example. There are \( 7^3 = 7 \times 7 \times 7 = 343 \) choices for (wife, sack, cat).

Example. There are \( 26^3 \times 10^3 = 17,576,000 \) license plates. Of these
\[
26 \times 25 \times 24 \times 10 \times 9 \times 8 = 11,232,000
\]
have distinct letters and digits (no repetition).

Permutations

A permutation of length \( r \) is a list \((x_1, \ldots, x_r)\) with distinct components (no repetition); that is, \( x_i \neq x_j \) when \( i \neq j \).

Examples. \((1, 2, 3)\) is a permutation of three elements; \((1, 2, 1)\) is a list, but not a permutation.

Counting Formulas. From \( n \) objects,
\[
\binom{n}{r} = \frac{n!}{r!(n-r)!}
\]
lists of length \( r \) and
\[
(n)_r := n \times (n-1) \times \cdots \times (n-r+1)
\]
permutations of length \( r \) may be formed.

Examples: There are \( 10^3 = 1000 \) three digit numbers of which \( (10)_3 = 10 \times 9 \times 8 = 720 \) list distinct digits.

Some Notation. Recall
\[
(n)_r = n \times (n-1) \times \cdots \times (n-r+1)
\]
positive integers \( n \) and \( r \).

Factorials: When \( r = n \), write
\[
n! = (n)_n = n \times (n-1) \times \cdots \times 2 \times 1.
\]
Conventions: \( (n)_0 = 1 \) and \( 0! = 1 \).

Notes a). The book only considers \( r = n \).
   b). \( (n)_r = 0 \) if \( r > n \).
   c). If \( r < n \), then
\[
n! = (n)_r (n-r)!
\]

Examples

Example. A group of 9 people may choose officers \((P, VP, S, T)\) in \((9)_4 = 3024\) ways.

Example. 7 books may be arranged in \( 7! = 5040 \) ways.

If there are 4 math books and 3 science books, then there are
\[
2 \times 4! \times 3! = 288
\]
arrangements in which the math books are together and the science books are together.
Combinations

A combination of size $r$ is a set $\{x_1, \ldots, x_r\}$ of $r$ distinct elements. Two combinations are equal if they have the same elements, possibly written in different orders.

*Example.* $\{1, 2, 3\} = \{3, 2, 1\}$, but $(1, 2, 3) \neq (3, 2, 1)$.

*Example.* How many committees of size 4 may be chosen from 9 people? Choose officers in two steps:

Choose a committee in $\binom{9}{4}$ ways.

Choose officers from the committee in $4!$ ways.

From the Basic Principle

$$\binom{9}{4} = 4! \times \binom{?}{?}.$$ 

So,

$$\binom{?}{4} = \frac{9!}{4!} = 126.$$

Binomial Coefficients

*Alternatively:*

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

The Binomial Theorem: For all $-\infty < x, y < \infty$,

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$ 

*Example.* When $n = 3$,

$$(x + y)^3 = x^3 + 3x^2y + 3xy^2 + y^3.$$ 

*Proof.* If

$$(x + y)^n = (x + y) \times \cdots \times (x + y)$$ 

is expanded, then $x^r y^{n-r}$ will appear as often as $x$ can be chosen from $r$ of the $n$ factors; i.e., in

$$\binom{n}{r}$$

ways.

Combinations Formula

From $n \geq 1$ objects,

$$\binom{n}{r} = \frac{1}{r!} (n)_r$$

combinations of size $r \leq n$ may be formed.

*Example.*

$$\binom{9}{4} = \frac{1}{4!} (9)_4 = 126.$$ 

*Proof.* Replace 9 and 4 by $n$ and $r$ in the example.

*Example: Bridge.* A bridge hand is a combination of $n = 13$ cards drawn from a standard deck of $N = 52$. There are

$$\binom{52}{13} = 635,013,559,600$$

such hands.

Binomial Identities

*Recall:*

$$(x + y)^n = \sum_{r=0}^{n} \binom{n}{r} x^r y^{n-r}.$$ 

*Examples.*

a). Setting $x = y = 1$,

$$\sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

b). Letting $x = -1$ and $y = 1$,

$$\sum_{r=0}^{n} \binom{n}{r} (-1)^r = 0$$

for $n \geq 1$. 
**Partitions**

**AKA Divisions**

**An Example**

Q: How many distinct arrangements can be formed from the letters MISSISSIPPI?

A: There are 11 letters which may be arranged in 

\[11! = 39,916,800\] 

ways, but this leads to double counting. If the 4 “S”s are permuted, then nothing is changed. Similarly, for the 4 “I”s and 2 “P”s. So, (*) the each configuration of letters 

\[4! \times 4! \times 2! = 1,152\] 

times and the answer is 

\[\frac{11!}{4! \times 4! \times 2!} = 34,650.\]

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**Example.** In the “MISSISSIPPI” Example, 11 positions, 

\[Z = \{1, 2, \cdots, 11\}\]

were partitioned into four groups of sizes 

- \(n_1 = 4\) “I”s
- \(n_2 = 1\) “M”s
- \(n_3 = 2\) “P”s
- \(n_4 = 4\) “S”s

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**Example.** In a bridge game, a deck of 52 cards is partitioned into four hands of size 13 each, one for each of South, West, North, and East.

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**Partitions**

**Definitions**

Let \(Z\) be a set with \(n\) elements. If \(r \geq 2\) is an integer, then an ordered partition of \(Z\) into \(r\) subsets is a list 

\[(Z_1, \cdots, Z_r)\]

where \(Z_1, \cdots, Z_r\) are mutually exclusive subsets of \(Z\) whose union is \(Z\); that is, 

\[Z_i \cap Z_j = \emptyset \text{ if } i \neq j\]

and 

\[Z_1 \cup \cdots \cup Z_r = Z.\]

Let 

\[n_i = \#Z_i,\]

the number of elements in \(Z_i\). Then 

\[n_1, \cdots, n_r \geq 0\]

and 

\[n_1 + \cdots + n_r = n.\]

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**The Partitions Formula**

Let \(n, r, \text{ and } n_1, \cdots, n_r\) be integers for which 

\[n, r \geq 1,\]

\[n_1, \cdots, n_r \geq 0,\]

\[n_1 + \cdots + n_r = n.\]

If \(Z\) is a set of \(n\) elements, then there are 

\[\binom{n}{n_1, \cdots, n_r} := \frac{n!}{n_1! \times \cdots \times n_r!}\]

ways to partition \(Z\) into \(r\) subsets \((Z_1, \cdots, Z_r)\) for which \(\#Z_i = n_i\) for \(i = 1, \cdots, r\).

**Example.**

\[\binom{11}{4, 1, 2, 4} = 34,650.\]

**Def.** Called multinomial coefficients
The Number of Solutions

If $n$ and $r$ are positive integers, how many integer solutions to the equations

$$n_1, \ldots, n_r \geq 0$$

$$n_1 + \cdots + n_r = n$$

are there?

**First Warm Up Example.** How many arrangements from $a$ A’s and $b$ B’s—for example, ABAAB)? There are

$$\binom{a + b}{a} = \binom{a + b}{b}$$

such, since an arrangement is determined by the $a$ places occupied by A.

A General Formula

If $n$ and $r$ are positive integers, then there are

$$\binom{n + r - 1}{r - 1} = \binom{n + r - 1}{n}$$

integer solutions to

$$n_1, \ldots, n_r \geq 0$$

$$n_1 + \cdots + n_r = n.$$ 

If $n \geq r$, then there are

$$\binom{n - 1}{r - 1}$$

solutions with

$$n_i \geq 1$$

for $i = 1, \ldots, r$.

The Number of Solutions

Continued

**Second Warm Up Example.** Suppose $n = 8$ and $r = 4$. Represent solutions by $o$ and ” + ” by |. For example,

$$ooo|oo|ooo$$

means

$$n_1 = 3,$$

$$n_2 = 2,$$

$$n_3 = 0,$$

$$n_4 = 3.$$ 

*Note: Only $r - 1 = 3$ ’s are needed.*

There are as many solutions as there are ways to arrange $o$ and |. By the last example, there are

$$\binom{8 + 3}{3} = \binom{11}{3} = 165$$

solutions.

Combinatorics

Summary

- Lists, permutations, and combinations.
- The Basic Principle
- Counting Formulas
  - Lists $n^r$
  - Permutations $(n)_r$
  - Combinations $\binom{n}{r}$
  - Partitions $\binom{n}{n_1, \ldots, n_r}$
  - Solutions $\binom{n+r-1}{r-1}$