Continuous Random Variables
Chapter 5
Important Concepts

- Distribution Functions.
- Density Functions.
- The Median and Other Quantiles.
- Expectation; the Mean and Variance
- Special Distributions
  - Uniform
  - Exponential
  - Normal
Continuous Random Variables

Introductory Example
Random Angles

For the Uniform Spinner,

\[ \Omega = (-\pi, \pi], \]

\[ P(a, b] = \frac{b - a}{2\pi} \]

for \(-\pi \leq a < b \leq \pi\). Let

\[ X(\omega) = \omega \]

\[ Y(\omega) = \tan(\omega) \]

for \(-\pi < \omega \leq \pi\). Then \(X\) and \(Y\) are random variables.

*Note: \( P[X = x] = 0 \) for all \( w \in \mathbb{R} \).*
Distributions Functions

Def. Given a model \((\Omega, P)\) and a RV

\[ X : \Omega \to \mathbb{R}, \]

the distribution function of \(X\) is defined by

\[ F(x) = P[X \leq x] \quad (*) \]

for \(-\infty < x < \infty\).

Example. For the random angle

\[ F(x) = \frac{1}{2} + \frac{x}{2\pi} \]

for \(-\pi < x \leq \pi\), since

\[ P[X \leq x] = P[-\pi < X \leq x] = \frac{x + \pi}{2\pi}. \]

Notation: Write \(X \sim F\) if \(X\) is a RV with DF \(F\); that is, if (*) holds.
Two Important Relations

If \( X \sim F \), then
\[
P[a < X \leq b] = F(b) - F(a)
\]
for all \(-\infty < a < b < \infty\), since
\[
F(b) = P[X \leq b] = P[X \leq a] + P[a < X \leq b] = F(a) + P[a < X \leq b]
\]
Also,
\[
P[X > b] = 1 - P[X \leq b] = 1 - F(b).
\]

**Example.** In the random angle example,
\[
F(x) = \frac{1}{2} + \frac{x}{2\pi},
\]
\[
F\left(\frac{2}{3}\pi\right) - F\left(\frac{1}{3}\pi\right) = \left(\frac{1}{2} + \frac{1}{3}\right) - \left(\frac{1}{2} + \frac{1}{6}\right) = \frac{1}{6}.
\]
Characteristic Properties

**Def.** A function $F : \mathbb{R} \rightarrow \mathbb{R}$ is *non-decreasing* if \( F(x) \leq F(y) \) whenever \( x \leq y \).

**Theorem.** A function $F : \mathbb{R} \rightarrow \mathbb{R}$ is the distribution function of some RV iff

\[
F(x) \leq F(y) \text{ whenever } x \leq y, \quad (1)
\]

\[
F(x) = \lim_{y \downarrow x} F(y) \text{ each all } x, \quad (2)
\]

\[
\lim_{x \to -\infty} F(x) = 0, \quad (3a)
\]

\[
\lim_{x \to \infty} F(x) = 1. \quad (3b)
\]

*Note:* Henceforth DF means a function satisfying (1), (2), and (3).
Proof

Outline-“Only If”

If $X \sim F$ and $x < y$, then

$$F(y) - F(x) = P[x < X \leq y] \geq 0,$$

establishing (1).

For (2), let

$$x_1 > x_2 > \cdots > x,$$

$$\lim_{n \to \infty} x_n = x,$$

$$B_n = \{\omega : X(\omega) \leq x_n\},$$

$$B = \{\omega : X(\omega) \leq x\},$$

so that

$$F(x) = P(B),$$

$$F(x_n) = P(B_n).$$

Then

$$B_1 \supseteq B_2 \supseteq \cdots$$
\[
\bigcap_{n=1}^{\infty} B_n = \{\omega : X(\omega) \leq x_n \text{ for all } n\} = B
\]

So,

\[
F(x) = P(B) = \lim_{n \to \infty} P(B_n) = \lim_{n \to \infty} F(x_n),
\]

by the Monotone Sequences Theorem.

The proof of (3) is similar. See Section 4.9.
The Discrete Case

If $X$ is discrete with range $\mathcal{X}$ and PMF $f$, then

$$ F(x) = P[X \leq x] = \sum_{y \in \mathcal{X}, y \leq x} f(y). $$

If also

$$ \mathcal{X} \subseteq \{ \cdots, -1, 0, 1, 2, \cdots \}, $$

then

$$ F(n) = \sum_{k \leq n} f(k) $$

and

$$ f(n) = F(n) - F(n - 1). $$
Densities

Def. A function $f : \mathbb{R} \to \mathbb{R}$ is a density if

\[ f(x) \geq 0, \text{ for all } x, \]

\[ \int_{-\infty}^{\infty} f(x)dx = 1. \]

If $X \sim F$ and $f$ is a density then $X$ and $F$ have density $f$ if

\[ F(x) = \int_{-\infty}^{x} f(y)dy \quad (4) \]

for all $-\infty < x < \infty$.

Theorem. If $f$ is any density, then (4) defines a DF.

Proof. Exercise.

Corollary. If $f$ is any density, then there is a RV $X$ with density $f$. 
Consequences Of

\[ F(x) = \int_{-\infty}^{x} f(y) \, dy \]  \hspace{1cm} (4)

If \( X \sim F \) with density \( f \), then

\[ P[a < X \leq b] = F(b) - F(a) = \int_{a}^{b} f(x) \, dx \]

for \(-\infty \leq a < b \leq \infty\).

If (4) holds, then

\[ f(x) = F'(x) = \frac{d}{dx} F(x) \]

at continuity points of \( f \).
Example
Uniform Distributions

If $-\infty < \alpha < \beta < \infty$, then

$$f(x) = \begin{cases} 1/(\beta - \alpha) & \text{if } \alpha < x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

is a density, since $f(x) \geq 0$ for all $x$ and

$$\int_{\infty}^{\infty} f(x)\,dx = \int_{\alpha}^{\beta} \frac{dx}{\beta - \alpha}$$

$$= \frac{1}{\beta - \alpha} (\beta - \alpha) = 1.$$  

Then

$$F(x) = \begin{cases} 0 & \text{if } x \leq \alpha \\ (x - \alpha)/(\beta - \alpha) & \text{if } \alpha < x \leq \beta \\ 1 & \text{if } x > \beta \end{cases}$$

**Example:** Random Angles: $\alpha = -\pi$ and $\beta = \pi$.  

Example

Exponential Distributions

If $0 < \lambda < \infty$, then

$$F(x) = \begin{cases} 
0 & \text{if } x \leq 0 \\
1 - e^{-\lambda x} & \text{if else}
\end{cases}$$

has density

$$f(x) = \begin{cases} 
0 & \text{if } -\infty < x < 0 \\
\lambda e^{-\lambda x} & \text{if } 0 \leq x < \infty.
\end{cases}$$

Notes

a). Derivative doesn’t exist when $x = 0$.

b). Doesn’t matter.
The Median and Other Quantiles

If \( X \sim F \), then and \( m \) for which

\[
F(m) = \frac{1}{2}
\]

is called a \textit{median} of \( X \) or \( F \). More generally, if \( 0 < p < 1 \), then any \( x \) for which

\[
F(x) = p
\]

is called a \( p^{th} \)-quantile or \( 100p^{th} \) percentile of \( X \) or \( F \). In terms of \( X \), the condition is

\[
P[X \leq x] = p.
\]
Example

If

\[ F(t) = \begin{cases} 
  0 & \text{if } \infty < t < 0 \\
  1 - e^{-\lambda t} & \text{if } 0 \leq t < \infty 
\end{cases} \]

then

\[ F(m) = \frac{1}{2} \]

iff

\[ 1 - e^{-\lambda m} = \frac{1}{2} \]

iff

\[ e^{-\lambda m} = \frac{1}{2} \]

iff

\[ e^{\lambda m} = 2 \]

iff

\[ m = \frac{1}{\lambda} \log(2), \]

where \( \log \) denotes natural logarithm.

**Example:** The half life of a radioactive substance.
Expectation

**Def.** If $X$ has density $f$, then the expectation of $X$ is defined by

$$E(X) = \int_{-\infty}^{\infty} x f(x) \, dx,$$

provided that the integral converges absolutely.

**Example:** *Uniform Distributions.* If

$$f(x) = \begin{cases} \frac{1}{\beta - \alpha} & \text{if } \alpha < x \leq \beta \\ 0 & \text{otherwise} \end{cases}$$

then

$$E(X) = \int_{\alpha}^{\beta} \frac{x \, dx}{\beta - \alpha}$$

$$= \frac{1}{2} \left[ \frac{x^2}{\beta - \alpha} \right]_{x=\alpha}^{\beta}$$

$$= \frac{1}{2} \left( \frac{\beta^2 - \alpha^2}{\beta - \alpha} \right)$$

$$= \frac{\alpha + \beta}{2},$$

since $\beta^2 - \alpha^2 = (\beta + \alpha)(\beta - \alpha)$. 
Properties of Expectation
As in the Discrete Case

**Transformations.** If

\[ Y = w(X), \]

then

\[ E(Y) = \int_{-\infty}^{\infty} w(x) f(x) dx, \]

provided that the integral converges absolutely.

**Linear Functions.**

\[ E(aX + b) = aE(X) + b, \]

since

\[
E(aX + b) = \int_{-\infty}^{\infty} (ax + b) f(x) dx
\]

\[
= a \int_{-\infty}^{\infty} x f(x) dx + b \int_{-\infty}^{\infty} f(x) dx
\]

\[ = aE(X) + b. \]
The Mean, Variance
And Standard Deviation

If $X$ has density $f$, then the expectation of $X$ is also called the mean of $X$ or $f$ and denoted by $\mu$, so that

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x) \, dx.$$  

The variance is

$$\sigma^2 = E[(X - \mu)^2];$$

and $\sigma$ is called the standard deviation.

As in the discrete case,

$$\sigma^2 = \int_{-\infty}^{\infty} (x - \mu)^2 f(x) \, dx.$$  

Also,

$$\sigma^2 = E(X^2) - \mu^2$$

and

$$E(X^2) = \int_{-\infty}^{\infty} x^2 f(x) \, dx.$$
An Example
A Uniform Distribution

If

\[ f(x) = \begin{cases} 
1 & \text{if } 0 \leq x \leq 1 \\
0 & \text{if otherwise} 
\end{cases} \]

then

\[ \mu = \int_0^1 x \, dx = \frac{1}{2} x^2 \bigg|_{x=0}^{x=1} = \frac{1}{2}, \]

\[ E(X^2) = \int_0^1 x^2 \, dx = \frac{1}{3} x^3 \bigg|_{x=0}^{x=1} = \frac{1}{3} \]

and

\[ \sigma^2 = E(X^2) - \mu^2 = \frac{1}{2} - \frac{1}{4} = \frac{1}{12}. \]

More generally, if

\[ X \sim \text{Unif}(\alpha, \beta), \]

then

\[ Var(X) = \frac{(\beta - \alpha)^2}{12}. \]
The Gamma Function

Let
\[ \Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} \, dx \]
for \( \alpha > 0 \). Then
\[ \Gamma(1) = \int_0^\infty e^{-x} \, dx = -e^{-x}|_{x=0}^\infty = 1, \]
and
\[ \Gamma(\alpha + 1) = \alpha \Gamma(\alpha) \]
for \( \alpha > 0 \). For, integrating by parts,
\[ \Gamma(\alpha) = \int_0^\infty x^\alpha e^{-x} \, dx \]
\[ = -x^\alpha e^{-x}|_{x=0}^\infty + \int_0^\infty \alpha x^{\alpha-1} e^{-x} \, dx \]
\[ = \alpha \Gamma(\alpha). \]
Then

\[ \Gamma(n) = (n - 1)\Gamma(n - 1) \]
\[ = (n - 1)(n - 2)\Gamma(n - 2) \]
\[ = \cdots \]
\[ = (n - 1)! \]

for integers \( n = 2, 3, \cdots \).
The Mean and Variance Of Exponential Distributions

If

\[
f(x) = \begin{cases} 
0 & \text{if } x < 0 \\
\lambda e^{-\lambda x} & \text{if } 0 \leq x < \infty 
\end{cases}
\]

then

\[
\mu = \int_{0}^{\infty} x \lambda e^{-\lambda x} \, dx
\]

\[
= \frac{1}{\lambda} \int_{0}^{\infty} x e^{-\lambda x} \, dx
\]

\[
= \frac{1}{\lambda} \Gamma(2)
\]

\[
= \frac{1}{\lambda}
\]
The Mean and Variance Of Exponential Distributions

Continued

Similarly,

\[ E(X^2) = \int_0^\infty x^2 \lambda e^{-\lambda x} \, dx \]

\[ = \frac{1}{\lambda^2} \int_0^\infty x^2 e^{-\lambda x} \, dx \]

\[ = \frac{1}{\lambda^2} \Gamma(3) \]

\[ = \frac{2}{\lambda^2} \]

and

\[ \sigma^2 = E(X^2) - \mu^2 = \frac{1}{\lambda^2}. \]