Multiple Integrals
An Interlude

If
\[-\infty < a < b < \infty,\]
\[-\infty < c < d < \infty,\]
and
\[f : [a,b] \times [c,d] \to \mathbb{R},\]
is sufficiently nice (e.g. continuous), then
\[\int_a^b \int_c^d f(x,y)dydx\]
is the limit of Riemann sums.

Extensions: Under mild technical conditions,
\[a, c \to -\infty,\]
\[b, d \to \infty.\]
**Bivariate Densities**

A function \( f : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a **bivariate density** if

\[ f(x, y) \geq 0, \]

and

\[ \int_{\mathbb{R}^2} f(x, y) dy dx = 1. \]

If \( f \) is a density, then JDRVs \( X \) and \( Y \) have **joint density** \( f \)

\[ P[(X, Y) \in C] = \int_C f(x, y) dy dx \]

for nice subsets \( C \subseteq \mathbb{R}^2 \).

**Example:** Uniform Distributions. If \( R \subseteq \mathbb{R}^2 \) and \( 0 < \alpha = \text{Area}(R) < \infty \), then

\[ f(x, y) = \frac{1}{\alpha} I_R(x, y) \]

is a density, called the **uniform density over** \( R \).

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**Marginal Densities**

If \( X \) and \( Y \) have joint density \( f \), then \( X \) and \( Y \) have individual (marginal) densities

\[ f_X(x) = \int_{-\infty}^{\infty} f(x, y) dy, \]

\[ f_Y(y) = \int_{-\infty}^{\infty} f(x, y) dx. \]

**Example:** Let

\[ D = \{ (x, y) : x^2 + y^2 \leq 1 \} \]

and

\[ f(x, y) = \frac{1}{\pi} 1_D(x, y). \]

If \( -1 < x < 1 \), then

\[ f_X(x) = \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \frac{1}{\pi} dy = \frac{2}{\pi} \sqrt{1-x^2}. \]

**Multivariate Densities** A function \( f : \mathbb{R}^m \rightarrow \mathbb{R} \) is a **m-variate density** if

\[ f(x) \geq 0, \]

and

\[ \int_{\mathbb{R}^m} f(x) dx = 1. \]

If \( f \) is a density, then JDRVs \( X_1, \ldots, X_m \) have **joint density** \( f \)

\[ P[X \in C] = \int_C f(x) dx \]

for nice subsets \( C \subseteq \mathbb{R}^m \).

**Marginal Densities:** If \( X_1, \ldots, X_j \) and \( Y_j, \ldots, Y_k \) have joint density \( f \), then \( X_1, \ldots, X_j \) have joint density

\[ f_X(x) = \int_{\mathbb{R}^{k-j}} f(x, y) dy. \]
Joint Distribution Functions

Def: If X and Y are JDRVs, then their joint distribution function is

\[ F(a, b) = P[X \leq a, Y \leq b]. \]

Marginal Distributions: Then

\[ F_X(a) = \lim_{b \to \infty} F(a, b), \]
\[ F_Y(b) = \lim_{a \to \infty} F(a, b). \]

Notes
a) Characteristic Properties
b) Higher Dimensions
c) Harder to Use
d) Mixed Distributions

Independence

JDRVs X and Y are independent if

\[ P[X \in A, Y \in B] = P[X \in A]P[Y \in B] \]

for all nice subsets \( A, B \subseteq \mathbb{R} \) (for example, intervals).

Conditions for Independence

DFs: X and Y are independent iff

\[ F(a, b) = F_X(a)F_Y(b) \]

for all \( a, b \in \mathbb{R} \).

Densities: If X and Y have individual densities \( f_X \) and \( f_Y \), then X and Y are independent iff X and Y have joint density

\[ f(x, y) = f_X(x)f_Y(y). \]

Example

If

\[ X \sim \text{Exp}(\lambda), \]
\[ Y \sim \text{Exp}(\lambda) \]

are independent, what is

\[ P[Y \geq 2X \text{ or } X \geq 2Y]. \]

Here

\[ f_X(z) = f_Y(z) = \lambda e^{-\lambda z} \]

for \( 0 \leq z < \infty \). So, \( f(x, y) = f_X(x)f_Y(y) = \lambda^2 e^{-\lambda(x+y)} \)

for \( 0 \leq x, y < \infty \) and \( f(x, y) = 0 \) for other \( x \) and \( y \). So,

\[ P[Y \geq 2X] = \int_0^\infty \left[ \int_0^{2x} \lambda^2 e^{-\lambda(x+y)} dy \right] dx \]
\[ = \int_0^\infty \left( \int_0^{\infty} \lambda^2 e^{-\lambda(x+y)} dy \right) dx \]
\[ = \int_0^\infty \lambda^2 \lambda e^{-\lambda x} dx \]
\[ = \frac{1}{3} \int_{x=0}^{\infty} e^{-3\lambda x} dx \]
\[ = \frac{1}{3}. \]

Similarly,

\[ P[X \geq 2Y] = \frac{1}{3}. \]

So,

\[ P[Y \geq 2X \text{ or } X \geq 2Y] = \frac{2}{3}. \]
Several Variables

$X_1, \ldots, X_m$ are independent if

$$P[X_1 \in A_1, \ldots, X_m \in A_m] = P[X_1 \in A_1] \times \cdots \times P[X_m \in A_m]$$

Note: Equivalent Conditions. For example,

$$f(x_1, \ldots, x_m) = f_1(x_1) \times \cdots \times f_m(x_m).$$

The Distribution of the Maximum

Example: $n$ light globes are placed in service at time $t = 0$ and allowed to burn continuously. Denote their lifetimes by $X_1, \ldots, X_n$ and suppose that

$$X_1, \ldots, X_n \sim \text{ind } F.$$  

If burned out globes are not replaced, then the room goes dark at time

$$Y = \max[X_1, \ldots, X_n],$$

the largest of $X_1, \ldots, X_n$.

The Distribution of $Y$: If (*) holds, then

$$G(y) := P[Y \leq y] = P[X_1 \leq y, \ldots, X_n \leq y] = P[X_1 \leq y] \times \cdots \times P[X_n \leq y] = F(y)^n.$$

So, if $F$ has density $f$, then $Y$ has density

$$g(y) = \frac{d}{dy}F(y)^n = nF(y)^{n-1}f(y).$$

Example: Revisited. If $n = 5$ and $F$ is exponential with $\lambda = 1$ per mo, then

$$F(t) = 1 - e^{-\lambda t},$$

and

$$G(t) = (1 - e^{-t})^5,$$

for $0 \leq t < \infty$. The probability that the room is still lighted after two months is

$$P[Y > 2] = 1 - G(2) = 1 - (1 - e^{-2})^5 = .5167.$$

Order Statistics

If

$$X_1, \ldots, X_n \sim \text{ind } F,$$

let

$$X_{(1)} \leq X_{(2)} \leq \cdots \leq X_{(n)}$$

by $X_1, \ldots, X_n$ in increasing order. Thus,

$$X_{(1)} = \min[X_1, \ldots, X_n],$$

$$\cdots,$$

$$X_{(n)} = \max[X_1, \ldots, X_n].$$

Notes

- Times that globes burn out in the example.
- Can find distributions.
- Section 6.6 and Problems 9 and 10,
**Convolution**

The Continuous Case

Let $X$ and $Y$ are independent with densities $f_X$ and $f_Y$, and let

$$Z = X + Y.$$ 

Then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx.$$ 

**Example:** If $X \sim \text{Unif}[0, 1]$ and $Y \sim \text{Unif}[0, 1]$, then

$$f_Z(z) = \min[z, 2-z]$$

for $0 \leq z \leq 2$ and $f_Z(z) = 0$ otherwise.

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In this case

$$f_X(z) = f_Y(z) = 1 \text{ for } 0 \leq z \leq 1,$$

$$f_X(z) = f_Y(z) = 0 \text{ otherwise}.$$ 

So, if $0 \leq z \leq 1$, for example, then

$$f_Z(z) = \int_{-\infty}^{\infty} f_X(x)f_Y(z-x)dx$$

$$= \int_0^z 1 \times 1 dx$$

$$= z.$$ 

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**Similarly**

Let

$$X_1, \ldots, X_n$$

be independent

and

$$Y = X_1 + \cdots + X_n.$$ 

If

$$X_i \sim \text{Gamma}(\alpha_i, \beta), \ i = 1, \cdots, n,$$

then

$$Y \sim \text{Gamma}(\alpha_1 + \cdots + \alpha_n, \beta).$$

If

$$X_i \sim \text{Normal}(\mu_i, \sigma^2_i), \ i = 1, \cdots, n,$$

then

$$Y \sim \text{Normal}(\mu, \sigma^2),$$

where

$$\mu = \mu_1 + \cdots + \mu_n,$$

$$\sigma^2 = \sigma^2_1 + \cdots + \sigma^2_n.$$ 

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**Conditional Distributions**

The Continuous Case

**Conditional Densities:** Let $X$ and $Y$ have joint density $f$. If $f_X(x) > 0$, then the conditional density of $Y$ given $X$ is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}.$$ 

As above, this is a density.

**Conditional Probability:** Write

$$P[Y \in B|X = x] = \int_B f_{Y|X}(y|x)dy.$$ 

**Notes:**

- New definition.
- $P[X = x] = 0$.
- Can reverse the roles of $X$ and $Y$. 
Example

If
\[ f(x, y) = \frac{2}{(1 + x + y)^2} \]
for \(0 \leq x, y < \infty\), then
\[ f_X(x) = \frac{1}{(1 + x)^2} \]
for \(0 \leq x < \infty\). So,
\[ f_{Y|X}(y|x) = \frac{2(1 + x)^2}{(1 + x + y)^3} \]
and
\[ P[Y > c|X = x] = \int_c^\infty \frac{2(1 + x)^2}{(1 + x + y)^3} dy \]
\[ = \frac{(1 + x)^2}{(1 + x + c)^3} \]

Bayes Theorem

In both cases (discrete and continuous),
\[ f(x, y) = f_{Y|X}(y|x) f_X(x) \]
if \( f_X(x) > 0 \). In the discrete case,
\[ f_Y(y) = \sum_{x \in X} f(x, y) \tag{\star} \]
and
\[ f_{X|Y}(x|y) = f(x, y)/f_Y(y) \]
when \( f_Y(y) > 0 \). In the continuous case, the sum in (\star) is replaced by an integral.

Mixed Distributions: One variable can be discrete and the other continuous.

Multivariate Extensions: \( X \) and/or \( Y \) can be vectors.

The Rule of Succession

Suppose
\[ X \sim \text{Unif}[0, 1], \]
and
\[ f_Y(y_1, \ldots, y_n|x) = x^{y_1 + \cdots + y_n} (1 - x)^{n - (y_1 + \cdots + y_n)} \]
for \( y_1, \ldots, y_n = 0 \) or 1. Then
\[ P[Y_1 = 1, \ldots, Y_n = 1] - f_Y(1, \ldots, 1) \]
\[ = \int_0^1 x^ndy \]
\[ = \frac{1}{n + 1} \]

If there were an \((n + 1)\)th \( Y \), then
\[ P[Y_{n+1} = 1|Y_1 = 1, \ldots, Y_{n+1} = 1] \]
\[ = \frac{P[Y_1 = 1, \ldots, Y_{n+1} = 1]}{P[Y_1 = 1, \ldots, Y_n = 1]} \]
\[ = \frac{1}{1(n + 2)} \]
\[ = \frac{n + 1}{n + 2} \]

Note: Depends (crucially) on the distribution of \( X \).
**Transformations**

Let

\[ m \geq 1, \]
\[ D \subseteq \mathbb{R}^m, \]

and let

\[ w : D \rightarrow \mathbb{R}^m, \]

be surjective (one-to-one). Write

\[ y = w(x) \]

as

\[ y_1 = w_1(x_1, \ldots, x_m), \]
\[ y_2 = w_2(x_1, \ldots, x_m), \]
\[ \ldots, \]
\[ y_m = w_m(x_1, \ldots, x_m), \]

**Jacobians**

Assuming that \( w \) is differentiable, let

\[ J_w(x) = \left| \det \begin{pmatrix} \frac{\partial w_1(x)}{\partial x_1} & \cdots & \frac{\partial w_1(x)}{\partial x_m} \\ \vdots & \ddots & \vdots \\ \frac{\partial w_m(x)}{\partial x_1} & \cdots & \frac{\partial w_m(x)}{\partial x_m} \end{pmatrix} \right|, \]

where

\[ x = (x_1, \ldots, x_m). \]

Or, briefly,

\[ J(x) = \left| \det \frac{\partial w(x)}{\partial x} \right|. \]

**Polar Coordinates**

Let

\[ m = 2, \]
\[ D = (0, \infty) \times [-\pi, \pi), \]

\[ y_1 = r \cos(\theta), \]
\[ y_2 = r \sin(\theta). \]

Then

\[ w : D \rightarrow \mathbb{R}^2 - \{0\} \]

\[ \frac{\partial w_1}{\partial r} = \cos(\theta), \]
\[ \frac{\partial w_2}{\partial r} = \sin(\theta), \]
\[ \frac{\partial w_1}{\partial \theta} = -r \sin(\theta), \]
\[ \frac{\partial w_2}{\partial \theta} = r \sin(\theta), \]

So,

\[ J(r, \theta) = \left| \det \begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -r \sin(\theta) & r \cos(\theta) \end{pmatrix} \right|. \]

That is,

\[ J(r, \theta) = r \cos^2(\theta) + r \sin^2(\theta) = r. \]
Change of Variables

Suppose that
\[ w : D \to \mathbb{R}^m \]
is differentiable and one-to-one and let \( v : E \to D \)
be the inverse function.

**Theorem.** Under technical conditions,
\[ \int_D g(x) dx = \int_E g[v(y)] J_v(y) dy. \]

**Corollary.**
\[ \int_{\mathbb{R}^2} g(x_1, x_2) dx_2 dx_1 \\
- \int_0^\infty \int_{-\pi}^{\pi} g(r \cos(\theta), r \sin(\theta)) r d\theta dr. \]

Transformations of RVs

Now let
\[ X_1, \ldots, X_m \sim f \text{ joint density}. \]
Let \( \mathbf{X} = (X_1, \ldots, X_m) \).

Suppose
\[ P[X \in D] = 1 \]
and let
\[ \mathbf{Y} = w(\mathbf{X}), \]
where
\[ w : D \to \mathbb{R}^m \subset \mathbb{R}^m. \]

That is,
\[ Y_1 = w_1(X_1, \ldots, X_m), \]
\[ Y_2 = w_2(X_1, \ldots, X_m), \]
\[ \ldots, \]
\[ Y_m = w_m(X_1, \ldots, X_m). \]

Example

If
\[ X_1, X_2 \sim \text{ind } \phi, \]
then
\[ f(x_1, x_2) = \frac{1}{\sqrt{2\pi}} e^{-x_1^2/2} \times \frac{1}{\sqrt{2\pi}} e^{-x_2^2/2} \]
\[ = \frac{1}{2\pi} e^{-x_1^2/2-x_2^2/2}. \]
\[ X_1 = R \cos(\Theta), \]
\[ X_2 = R \sin(\Theta), \]
then
\[ g(r, \theta) = f[r \cos(\theta), r \sin(\theta)] r \]
\[ = \frac{r}{2\pi} e^{-r^2/2} \]
for \( 0 < r < \infty \) and \( -\pi < \theta \leq \pi \).

Suppose that
\[ w \text{ is surjective,} \]
\[ v = w^{-1} \text{ is smooth.} \]

**Theorem.** \( \mathbf{Y} \) has density
\[ g(y) = f[v(y)] J_v(y) 1_E(y). \]

**Corollary**
\[ g_1(y_1) = \int_{\mathbb{R}^n} g(y_1, z) dz. \]

**Corollary.** If
\[ X_1, X_2 \sim f, \]
\[ X_1 = R \cos(\Theta), \]
\[ X_2 = R \sin(\Theta), \]
then
\[ g(r, \theta) = f[r \cos(\theta), r \sin(\theta)] r \]
for \( 0 < r < \infty \) and \( -\pi < \theta \leq \pi \).
So,

\[ g_1(r) = \int_{-\pi}^{\pi} \frac{r}{2\pi} e^{-\frac{\theta^2}{2}} d\theta = r e^{-\frac{\pi^2}{2}} \]

for \(0 < r < \infty\); and

\[ g_2(\theta) = \int_{0}^{\infty} \frac{r}{2\pi} e^{-\frac{\theta^2}{2r^2}} dr = \frac{1}{2\pi} e^{-\frac{\pi^2}{2r^2}} \bigg|_{r=0}^{\infty} = r e^{-\frac{\pi^2}{2}} \]

and

\( R \) and \( \Theta \) are independent.