Math/ Stat 425 Problem Set 6 Solutions

2. The mean and variance of an \( X_i \) are
\[
\mu = \int xf(x)dx = \frac{3}{4} \int_{-1}^{1} x(1 - x^2)dx = 0,
\]
by symmetry, and
\[
\sigma^2 = \frac{3}{4} \int_{-1}^{1} x^2(1 - x^2)dx = \frac{3}{2} \int_{0}^{1} x^2(1 - x^2)dx = \frac{3}{2} \left[ \frac{1}{3} x^3 - \frac{1}{5} x^5 \right]_{x=0}^{1} = \frac{1}{5},
\]
again using the symmetry. So, the mean and variance of \( S \) are
\[
n\mu = 80 \times 0 = 0 \quad \text{and} \quad n\sigma^2 = 80/5 = 16;
\]
and \( S \) is approximately normal by the Central Limit Theorem. So,
\[
P[S > 3] = 1 - P[S \leq 3] \approx 1 - \Phi\left(\frac{3}{4}\right) = .2266,
\]
and
\[
c = \sqrt{n\sigma^2} \times 1.645 = 4 \times 1.645 = 6.58.
\]

4. Let \( Y_i \) denote the state’s gain of the \( i^{th} \) day, so that \( S = Y_1 + \cdots + Y_{100} \). Now, \( Y_i = 1000 - 500X_i \) where \( X_i \) is the number of winners on the \( i^{th} \) day and \( X_i \sim \text{Poisson}(\lambda = 1) \). So, \( E(Y_i) = 500 \) and \( E(S) = 100 \times 500 = 50,000 \). The variance of an \( X_i \) is one. So, the variance of a \( Y_i \) is \((500)^2\), the variance of \( S \) is \( 100 \times (500)^2 \), and the standard deviation of \( S \) is \( 10 \times 500 = 5000 \). So,
\[
c = 5000 \times 1.96 = 9800.
\]

6. Generate \( 2N \) independent uniforms \( U_1, U_2, \cdots, U_{2N} \), and let
\[
\hat{p} = \frac{\#\{k \leq N : U_{2k-1}^2 + U_{2k}^2 \leq 1\}}{N}.
\]
Then the standard deviation of \( \hat{p} \) is \( \sqrt{pq/N} \leq 1/(2\sqrt{N}) \), where \( p \) is the desired probability. To make this less than \( .01 \) we need \( N \geq 2500 \).

8. The mean of an \( X_k \) is zero, and the variance is \( 3\tau^2/8 \); and the covariance between \( X_j \) and \( X_k \) is zero if \( |j - k| > 2 \). Let \( S_n = X_1 + \cdots + X_n \). Then \( E(S_n) = 0 \), and the variance of \( S_n \) is
\[
E(S_n^2) = \sum_{i=1}^{n} \sum_{j=1}^{n} C(X_i, X_j) \leq 5n \times \frac{3\tau^2}{8}.
\]
So,
\[
E(\bar{X}_n^2) = \frac{1}{n^2} E(S_n^2) \leq \frac{15\tau^2}{8n} \rightarrow 0
\]
as \( n \rightarrow \infty \). So, \( \bar{X}_n \) converges to 0 in mean square and, therefore, also in probability.