Confidence Intervals

Model: $X_1, \cdots, X_n$ have a joint distribution depending on $\theta$ (and possibly nuisance parameters).

Level 1 - $\alpha$ Confidence Intervals: $[L, U]$, where

\[
L = L(X_1, \cdots, X_n)
\]

\[
U = U(X_1, \cdots, X_n)
\]

and

\[
P[L \leq \theta \leq U] \geq 1 - \alpha
\]

for all $\theta$ (and other unknowns, if present).

Note: Often $\alpha = .1, .05, \text{or}. .01$.

Some Adjectives: Exact, if $= 1 - \alpha$; Conservative, otherwise. Asymptotic or Approximate.

Sampling From A Normal Distribution

Known $\sigma^2$

Suppose

\[
X_1, \cdots, X_n \sim_{ind} \text{Normal}(\theta, \sigma^2)
\]

where $\theta$ is unknown and $\sigma^2$ is known. Then, the MLE is $\bar{X}$ and

\[
\bar{X} \sim \text{Normal}(\theta, \frac{\sigma^2}{n}).
\]

So,

\[
Z = \frac{\bar{X} - \theta}{\sigma/\sqrt{n}} \sim \Phi
\]

and, therefore,

\[
P[-1.96 \leq Z \leq 1.96] = \Phi(1.96) - \Phi(-1.96) = .95.
\]

Now,

\[
-1.96 \leq \frac{\bar{X} - \theta}{\sigma/\sqrt{n}} \leq 1.96
\]

iff

\[
-1.96 \frac{\sigma}{\sqrt{n}} \leq \bar{X} - \theta \leq 1.96 \frac{\sigma}{\sqrt{n}}
\]

iff

\[
\bar{X} - 1.96 \frac{\sigma}{\sqrt{n}} \leq \theta \leq \bar{X} + 1.96 \frac{\sigma}{\sqrt{n}}.
\]

Let

\[
L, U = \bar{X} \pm 1.96 \frac{\sigma}{\sqrt{n}}.
\]

Then

\[
P[L \leq \theta \leq U] = P[-1.96 \leq Z \leq 1.96] = .95
\]

and $[L, U]$ is a level .95 confidence interval.

More Generally

If

\[
\hat{\theta} \sim \text{Normal}(\theta, \sigma_{\hat{\theta}}^2)
\]

where $\sigma_{\hat{\theta}}^2$ is known, then

\[
Z = \frac{\hat{\theta} - \theta}{\sigma_{\hat{\theta}}} \sim \Phi
\]

and

\[
[L, U] = [\hat{\theta} - 1.96 \sigma_{\hat{\theta}}, \hat{\theta} + 1.96 \sigma_{\hat{\theta}}]
\]

is a level .95 confidence interval.

Note: For a level 1 - $\alpha$ confidence interval, replace 1.96 with $\Phi^{-1}(1 - .5\alpha)$. 
**Unknown $\sigma^2$**

Then

$$\sigma^2 \hat{X} = \frac{\sigma^2}{n}$$

is unknown. Let

$$\hat{\sigma}^2 \hat{X} = \frac{S^2}{n},$$

where

$$S^2 = \left( \frac{1}{n-1} \right) \sum_{i=1}^{n} (X_i - \bar{X})^2$$

and

$$T = \frac{\bar{X} - \theta}{\hat{\sigma} \sqrt{n}} = \frac{\bar{X} - \theta}{S / \sqrt{n}}.$$ 

Then

$$T \sim t_{n-1},$$

the $t$-distribution on $n-1$ degrees of freedom.

Given $0 < \alpha < 1$, let $c = t_{n-1,1-\frac{1}{2}\alpha}$, the upper $(1 - \frac{1}{2}\alpha)^{th}$ quantile of the $t$ distribution. Then

$$P[-c \leq T \leq c] = P[T \leq c] - P[T \leq -c]$$

$$= (1 - .5\alpha) - .5\alpha = 1 - \alpha$$

and

$$-c \leq T \leq c \quad \text{iff} \quad \bar{X} - \frac{cS}{\sqrt{n}} \leq \theta \leq \bar{X} + \frac{cS}{\sqrt{n}}$$

as above. Let

$$L, U = \bar{X} \pm \frac{cS}{\sqrt{n}}$$

Then

$$P[L \leq \theta \leq U] = P[-c \leq T \leq c] = 1 - \alpha,$$

and $[L, U]$ is a level $1 - \alpha$ confidence interval.

**Example: A Soporific Drug.** Average hours sleep was recorded for several subjects to establish a baseline; then they started taking a drug, and average hours sleep was recorded again. The average additional hours sleep for ten subjects were

0.7, -1.6, -0.2, -1.2, -0.1, 3.4, 3.7, 0.8, 0.0, 2.0.

Here

$$\bar{X} = .750, \ S^2 = 3.201, \ n = 10,$$

and

$$c = 2.626 \text{ for } \alpha = .05.$$ 

So,

$$\bar{X} \pm \frac{cS}{\sqrt{n}} = .75 \pm 1.28$$

is a level .95 confidence interval

**Assumptions:** Sample from Normal($\theta, \sigma^2$).

**Notes and Terminology**

**Standardization:**

$$Z = \frac{\hat{\theta} - \theta}{\sigma_{\hat{\theta}}}.$$ 

**Studentization:**

$$T = \frac{\hat{\theta} - \theta}{\hat{\sigma}_{\hat{\theta}}}.$$ 

**Pivotal Quantities:** In the normal case, the distributions of $Z$ and $T$ are known.

**Historical Interest:** Data Set.
Confidence Intervals for $\sigma^2$

Let

$$G_r = \text{The } \chi^2_r \text{ DF},$$

so that

$$\frac{(n-1)S^2}{\sigma^2} \sim G_{n-1}.$$ 

Given $0 < \alpha < 1$, let

$$G_{n-1}(a) = \frac{1}{2} \alpha,$$

$$G_{n-1}(b) = 1 - \frac{1}{2} \alpha,$$

so that $G_{n-1}(b) - G_{n-1}(a) = 1 - \alpha$. Then

$$P[a \leq \frac{(n-1)S^2}{\sigma^2} \leq b] = G_{n-1}(b) - G_{n-1}(a) = 1 - \alpha.$$

Recall:

$$P[a \leq \frac{(n-1)S^2}{\sigma^2} \leq b] = 1 - \alpha.$$ 

Now

$$a \leq \frac{(n-1)S^2}{\sigma^2} \leq b \quad \text{iff} \quad \frac{(n-1)S^2}{b} \leq \sigma^2 \leq \frac{(n-1)S^2}{a}.$$ 

Let

$$L = \frac{(n-1)S^2}{b} \quad \text{and} \quad U = \frac{(n-1)S^2}{a}.$$ 

Then $[L, U]$ is a level $1 - \alpha$ confidence interval for $\sigma^2$.

Example: The Soporific Drug. If $n = 10$ and $\alpha = .05$, then $a = 2.7$ and $b = 19$. So, if $S^2 = 3.201$, then

$$L = \frac{9 \times 3.201}{19} = 1.52 \quad \text{and} \quad U = 10.7.$$ 

Robustness

Suppose now

$$X_1, \ldots, X_n \sim^{ind} F,$$

where $F$ has mean $\mu$ and variance $\sigma^2$, but is not normal.

??Sensitivity to Assumptions??

- If $r$ is large, then $T \approx \Phi$.
- If $F$ is continuous and symmetric, then the confidence intervals for $\mu$ are approximately valid for large $n$, $n \geq 50$.
- Confidence intervals for $\sigma^2$ are sensitive to the normality assumption.

Interpretation of Confidence Intervals

Correct: If the experiment were repeated many times, then a level $(1 - \alpha)$ confidence interval would cover the true value of $\theta$ is (approximately) $100(1 - \alpha)$ % of the replications.

Note: This is NOT to say that the conditional probability that $L \leq \theta \leq U$, given the data is $1 - \alpha$. To assign a conditional probability to $\theta$, an unconditional distribution has to be specified (as in the Bayesian approach).
**Credible Intervals**

**Def:** Suppose now that $\theta$ has a prior density, $g$ say, so that the posterior density is

$$g(\theta|x) = L(\theta|x)g(\theta),$$

where $L(\theta|x)$ is the likelihood function. Then $[L, U]$ is a level $1 - \alpha$ credible interval if

$$P[L \leq \theta \leq U|x] = 1 - \alpha. \quad (*)$$

**Note:** Note the left side of (*) is typically a subjective probability; it is the analyst’s probability, as opposed to the probability.

**Example**

Suppose

$$X_1, \ldots, X_n \sim \text{ind Normal} (\theta, \sigma^2),$$

$$\theta \sim \text{Normal} (\mu, \tau^2),$$

where $\sigma^2$ is known. Then

$$\theta|x \sim \text{Normal} (\mu', \tau'^2),$$

where

$$\mu' = \frac{\sigma^2 \mu + n\tau^2 \bar{x}}{\sigma_n^2 \tau^2},$$

$$\tau'^2 = \frac{\sigma^2 \tau^2}{\sigma^2 + n\tau^2}.$$