Stopping Times and The Strong Markov Property
April 16-18, 2001

Let $\mathcal{B}_t$, $0 \leq t < \infty$, denote standard Brownian motion, defined on a probability space $(\Omega, \mathcal{A}, P)$; and let

$$
\mathcal{B}_t = \sigma\{\mathcal{B}_s : 0 \leq s \leq t\},
$$

$$
\mathcal{B}_t^+ = \bigcap_{u > t} \mathcal{B}_u.
$$

for $0 \leq t < \infty$. Thus, $\mathcal{B}_t = \bigvee_J \sigma\{\mathcal{B}_s : s \in J\}$, where the join extends of all finite subsets $J \subseteq [0, t]$.

**Lemma 1.** For a fixed $t \in [0, \infty)$, let

$$
Z^t_s = \mathcal{B}_{t+s} - \mathcal{B}_t
$$

for $0 \leq s < \infty$. Then $Z^t_s$, $0 \leq s < \infty$ is again a standard Brownian motion; and (the sigma algebra generated by) $Z^t_s$, $0 \leq s < \infty$ is independent of $\mathcal{B}^+_t$.

**Proof.** It is clear that $Z^t_s$, $0 \leq s < \infty$, is a Gaussian process with continuous sample paths and mean 0. If $0 < s < u < \infty$, then

$$
E(Z^t_s Z^t_u) = E([Z^t_s]^2) + E\{Z^t_s [\mathcal{B}_{t+u} - \mathcal{B}_{t+s}]\} = s,
$$

so that $Z^t_s$, $0 \leq s < \infty$, is standard Brownian motion.

For the independence, let $\mathcal{C}_t = \sigma\{Z^t_s : 0 \leq s < \infty\}$. Then independence means that

$$
P(B \cap C) = P(B)P(C)
$$

(1)

for $B \in \mathcal{B}_t^+$ and $C \in \mathcal{C}_t$. It is first shown that $\mathcal{B}_t$ and $\mathcal{C}_t$ are independent. For this is suffices to verify (1) for sets of the form

$$
B = \{[\mathcal{B}_{t_1}, \ldots, \mathcal{B}_{t_m}] \in E\},
$$

$$
C = \{[Z^t_{s_1}, \ldots, Z^t_{s_n}] \in F\},
$$

where $0 \leq t_1 < \cdots < t_m$ and $0 < s_1 < \cdots < s_n < \infty$, $E \subseteq \mathbb{R}^m$ and $F \subseteq \mathbb{R}^n$ are Borel sets. In this case (1) is clear in this case, since $\mathcal{B}_s$, $0 \leq s < \infty$, has independent increments.

Next, suppose that $B \in \mathcal{B}_t^+$, so that $B \in \mathcal{B}_t'$ for all $t' > t$. If $F \subseteq \mathbb{R}^n$ is continuity set of $[Z^t_{s_1}, \ldots, Z^t_{s_n}]$, then

$$
1_F[Z^t_{s_1}, \cdots, Z^t_{s_n}] \rightarrow 1_F[Z^t_{s_1}, \cdots, Z^t_{s_n}] \text{ w.p.} 1
$$

1
as \( t' \downarrow t \) and, therefore,

\[
P[B \cap \{Z^t_{s_1}, \cdots, Z^t_{s_n} \in F\}] = \lim_{t' \downarrow t} P[B \cap \{Z^t_{s_1}, \cdots, Z^t_{s_n} \in G\}]
\]

\[
= \lim_{t' \downarrow t} P(B)P\{Z^t_{s_1}, \cdots, Z^t_{s_n} \in G\}
\]

\[
= P(B)P(C),
\]

establishing (1). \( \diamondsuit \)

**Notation:** Now let \( \mathcal{A}_t \subseteq \mathcal{A} \), \( 0 \leq t < \infty \), be a filtration (increasing family of sigma-algebras) for which

- \( \mathcal{B}_t \) is \( \mathcal{A}_t \)-measurable
- \( \mathcal{A}_t \) is independent of \( \mathcal{B}_{t+s} - \mathcal{B}_t \), \( 0 \leq s < \infty \)

for all \( 0 \leq t < \infty \). The filtration is said to be right continuous if \( \mathcal{A}_t = \cap_{u \geq t} \mathcal{A}_u = \mathcal{A}_{t^+} \), say, for all \( t \). For example, \( \mathcal{B}_t \) is right continuous.

**Exercise:** If \( \mathcal{C} \) is independent of \( \mathcal{B}_t \), \( 0 \leq t < \infty \), then \( \mathcal{A}_t = \mathcal{C} \vee \mathcal{B}_t^+ \) has the desired properties.

An extended random variable \( \tau : \Omega \rightarrow [0, \infty] \) is said to be a stopping time with respect to \( \mathcal{A}_t \), \( 0 \leq t < \infty \) iff

\[
\{\tau \leq t\} \in \mathcal{A}_t
\]

for all \( 0 \leq t < \infty \). If \( \tau \) is a stopping time, then \( \{\tau < t\} = \bigcup_{n=1}^{\infty} \{\tau \leq t - n^{-1}\} \in \mathcal{A}_t \) for all \( t \). Conversely if the filtration is right continuous and \( \{\tau < t\} \in \mathcal{A}_t \) for all \( t \), then \( \{\tau \leq t\} = \bigcap_{n=1}^{\infty} \{t < t + n^{-1}\} \in \mathcal{A}_{t^+} = \mathcal{A}_t \) for all \( t \), so that \( \tau \) is a stopping time.

For examples, let \( B \subseteq \mathbb{R} \). Then

\[
\tau = \inf\{t \geq 0 : \mathcal{B}_t \in B\}
\]

is called the time of first entry into \( B \).

**Proposition 1.** The first entry time \( \tau \) is a stopping time if either \( B \) is closed or \( B \) is open and \( \mathcal{A}_t \), \( 0 \leq t < \infty \) are right continuous.

**Proof.** If \( F \) is closed and \( t \geq 0 \), let \( F_0 \) be a countable dense subset of \( F \). Then

\[
\{\tau \leq t\} = \{ \inf_{s \in [0,t] \cap \mathbb{R}^\#} \inf_{x \in F_0} |B_s - x| = 0 \} \in \mathcal{A}_t,
\]

where \( \mathbb{R}^\# \) denotes the rational numbers. If \( B \) is open, \( \mathcal{A}_t \) are right continuous, and \( t > 0 \), then

\[
\{\tau < t\} = \bigcup_{s \in [0,t] \cap \mathbb{R}^\#} \{\mathcal{B}_s \in B\} \in \mathcal{A}_t,
\]

so that \( \tau \) is a stopping time. \( \diamondsuit \)
Example. If $0 \leq a, b < \infty$, then

$$
\tau_b = \inf \{ t \geq 0 : \mathcal{B}_t > b \},
\tau_{a,b} = \inf \{ t \geq 0 : \mathcal{B}_t \notin [-a, b] \}
$$

are stopping times. Moreover, the strict inequalities could be replaced by weak one. \hfill \diamond

The pre-$\tau$ Sigma Algebra. If $\tau$ is a stopping time, let

$$
\mathcal{A}_\tau = \{ A \in \mathcal{A} : A \cap \{ \tau \leq t \} \in \mathcal{A}_t, \text{ for all } 0 \leq t < \infty \}. 
$$

Remarks. As in the discrete case:

a) $\mathcal{A}_\tau$ is sigma-algebra.
b) Constants are stopping times, and the notation $\mathcal{A}_\tau$ is unambiguous.
c) If $\eta$ and $\tau$ are stopping times, then so are $\eta \land \tau$ and $\eta \lor \tau$.
d) If $\eta$ and $\tau$ are stopping times for which $\eta \leq \tau$, then $\mathcal{A}_\eta \subseteq \mathcal{A}_\tau$.
e) $\tau$ is $\mathcal{A}_\tau$-measurable.

Lemma 2. If $\tau$ is any stopping time, then there are stopping times $\tau_n$ for which:

$$
\tau_n \in \{ 0, \frac{1}{2^n}, \frac{2}{2^n}, \cdots, \infty \}
$$

$$
\tau_n = \tau \text{ on } \{ \tau = \infty \},
0 \leq \tau_n - \tau \leq \frac{1}{2^n} \text{ on } \{ \tau < \infty \};
$$

for all $n$. Further, if $\mathcal{A}_t$, $0 \leq t < \infty$, are right continuous, then

$$
\mathcal{A}_\tau = \bigcap_{n=1}^{\infty} \mathcal{A}_{\tau_n}.
$$

Proof. Let

$$
\tau_n = \sum_{k=1}^{\infty} k \frac{1}{2^n} 1 \{ \frac{k}{2^n} \leq \tau \leq \frac{k+1}{2^n} \} + \infty 1 \{ \tau = \infty \}.
$$

Then each $\tau_n = \infty$ whenever $\tau = \infty$, $0 \leq 0 \leq \tau_n - \tau \leq 1/2^n$ on $\{ \tau < \infty \}$, and (2) holds. Next, it is shown that each $\tau_n$ is a stopping time. If $t \geq 0$, let $k = \lfloor 2^n t \rfloor$. Then

$$
\{ \tau_n \leq t \} = \{ \tau \leq \frac{k}{2^n} \} \in \mathcal{A}_{\frac{k}{2^n}} \subseteq \mathcal{A}_t,
$$

as required.
For the final assertion, it is clear that \( A_r \subseteq A_{\tau_n} \) for all \( n \). If \( A \in \bigcap_{n=1}^{\infty} A_{\tau_n} \) and \( t \geq 0 \), then
\[
A \cap \{ \tau < t \} = A \cap \bigcup_{n=1}^{\infty} \{ \tau_n < t \} \in A_t
\]
and, therefore,
\[
A \cap \{ \tau \leq t \} = \bigcap_{n=1}^{\infty} \{ \tau + \frac{1}{n} < t \} \in A_{t+} = A_t.
\]

\( \diamond \)

**The Stopped Process:** Let
\[
\mathbb{I}_B(\omega) = \begin{cases} 
\mathbb{I}_B(\omega) & \text{if } \tau(\omega) < \infty, \\
-\infty & \text{if } \tau(\omega) = \infty.
\end{cases}
\]

**Lemma 4.** If \( \tau \) is a stopping time and \( A_t \), \( 0 \leq t < \infty \), are right continuous, then \( \mathbb{I}_B \) is \( A_\tau \)-measurable.

**Proof.** \( \cdots \)

**The Strong Markov Property.** Let \( \tau \) be a stopping time that is finite w.p.1; and let
\[
Z_s = \mathbb{I}_B_{t+s} - \mathbb{I}_B_t, \ 0 \leq s < \infty
\]
if \( \tau < \infty \) and \( Z_s \equiv 0 \) otherwise. Then \( Z_s \), \( 0 \leq s < \infty \), is standard Brownian motion and is independent of \( A_\tau \).

**Proof.** Suppose first that there is a countable set \( C \) for which \( \tau \in C \). If \( A \in A_\tau \) and \( 0 \leq s_1 < \cdots < s_m < \infty \), then
\[
P[A \cap \{ [Z_{s_1}, \cdots, Z_{s_m}] \in F \}] = \sum_{t \in C} P[A \cap \{ \tau = t \} \cap \{ [Z_{s_{t}}, \cdots, Z_{s_{t}}] \in F \}]
\]
\[
= \sum_{t \in C} P[A \cap \{ \tau = t \}] P[[B_{s_{t}}, \cdots, B_{s_{t}}] \in F]]
\]
\[
= P(A) P[[B_{s_{1}}, \cdots, B_{s_{m}}] \in F],
\]
by Lemma 1. So, the result is true in this case. For the general case, construct \( \tau_n \) as in Lemma 2; let \( F \) be a continuity set for \( [B_{s_{1}}, \cdots, B_{s_{m}}] \); and let
\[
Z_{n,s} = \mathbb{I}_B_{s_n+s} - \mathbb{I}_B_{s_n},
\]
If \( A \in A_\tau \), then \( A \in A_{\tau_n} \) for all \( n \) and, therefore,
\[
P[A \cap \{ [Z_{s_1}, \cdots, Z_{s_m}] \in F \}] = \lim_{n \to \infty} P[A \cap \{ [Z_{n,s_1}, \cdots, Z_{n,s_m}] \in F \}]
\]
\[
= \lim_{n \to \infty} P(A) P[[Z_{n,s_1}, \cdots, Z_{n,s_m}] \in F]
\]
\[
= P(A) P[[Z_{n,s_1}, \cdots, Z_{n,s_m}] \in F].
\]

A Reflection Principle. In the following, let \( \Phi_{m,\Sigma} \) denote the \( m \) variate normal distribution with mean 0 and covariance matrix \( \Sigma \).
Corollary. If $\tau$ is a stopping time for which $\tau \leq t_0 < \infty$ (everywhere), then

$$P[B_{t_0} - B_{\tau} \leq x | A_{\tau}] = \Phi_{1, \tau-t_0}(x).$$

Proof. Given $\epsilon > 0$, let $m = \lfloor 1/\epsilon^2 \rfloor$ and $\Sigma = \epsilon[i \wedge j : i, j = 1, \ldots, m]$. Then $\Phi_m, \Sigma$ is a regular conditional distribution for $B_{\tau+\epsilon i} - B_{\tau}, i = 1, \ldots, m$, by the Strong Markov Property. Let $i^*$ be the least $i$ for which $\tau + \epsilon i \geq t_0$. If $g$ is a bounded continuous function, then

$$E[g(B_{\tau+\epsilon i^*} - B_{\tau}) | A_{\tau}] = \int_{\mathbb{R}^m} g(y_{i^*}) \Phi_m, \Sigma \{dy_1, \ldots, dy_m\}$$

$$= \int_{\mathbb{R}} g(y) \Phi_{1, \epsilon i^*}.$$ 

Letting $\epsilon \downarrow 0$ and using the Dominated Convergence Theorem on the conditional expectation, then yields

$$E[g(B_{\tau+\epsilon i^*} - B_{\tau}) | A_{\tau}] = \int_{\mathbb{R}} g(y) \Phi_{1, \tau-t_0} \{dy\},$$

and the Corollary follows. $\diamond$

The Distribution of the Maximum. Now let

$$M_t = \max_{0 \leq s \leq t} B_s,$$

$$\tau_c = \inf\{t \geq 0 : B_t \geq c\}.$$

Then $M_t$ is a random variance; $\tau$ is a stopping time; and

$$P\{M_t \geq c\} = P\{\tau_c \leq c\}.$$

Theorem: The Distribution of The Maximum. For $0 < c, t < \infty$,

$$P[M_t \geq c] = 2[1 - \Phi(\frac{c}{\sqrt{t}})].$$

Proof. The left side is

$$P[\tau_c \leq t] = P[\tau_c \leq t, B_t \geq c] + P[\tau_c < t, B_t < c]$$

Here

$$P[\tau_c \leq t, B_t \geq c] = P[B_t \geq c] = 1 - \Phi(\frac{c}{\sqrt{t}}).$$

Next, since $B_{\tau_c} = c$, 

5
\[ P[\tau_c < t, |B_t| < c] = P[\tau_c < t, |B_t - B_{\tau_c}| < 0] \]
\[ = \int_{\tau_c < t} \Phi_{1,t-\tau_c}(0) dP \]
\[ = \int_{\tau_c < t} [1 - \Phi_{1,t-\tau_c}(0)] dP \]
\[ = P[\tau_c < t, |B_t - B_{\tau_c}| > 0] \]
\[ = P[|B_t| > c] \]
\[ = 1 - \Phi\left(\frac{c}{\sqrt{t}}\right), \]

and the theorem follows.

**Corollary 1.**
\[ P[\tau_c \leq t] = 2\Phi\left(\frac{c}{\sqrt{t}}\right) - 1. \]

**Corollary 2.** \( E(\tau_c) = \infty. \)

*An Optional Stopping Theorem.* There is also an optional stopping Theorem for Brownian motion.

**Lemma 4.** If \( \tau \) is a non-negative, integrable random variable, then \( M^*_\tau = \max_{s \leq \tau} |B_s| \) is integrable.

**Proof.** \( M_\tau \) is measurable, since the maximum may be restricted to rational values. For the integrability,
\[ P[M_\tau > x] \leq P \max_{s \leq x} |B_s| > x] + P[\tau > x] \leq 4[1 - \Phi(\sqrt{x})] + P[\tau > x] \]
for \( x > 0 \), and the right side is integrable. \( \diamond \)

**Theorem:** Wald’s Lemmas. If \( \tau \) is a stopping time for which \( E(\tau) < \infty \), then
\[ E[|B_\tau|] = 0, \]
\[ E[|B_\tau|^2] = E(\tau). \]

**Proof.** Construct \( \tau_n \) as in Lemma 2. Then \( \tau_n \leq \tau + 1 \), so that \( E(\tau_n) < \infty \). So, \( E(|B_{\tau_n}|) = 0 \) for each \( n \), by Wald’s Lemma in the discrete case. Moreover,
\[ |B_{\tau_n}| \leq \max_{s \leq \tau+1} |B_s|, \]
which is integrable by the last lemma. So, \( E(B_\tau) = \lim_{n \to \infty} E(B_{\tau_n}) = 0. \) This establishes the first assertion, and the second may be similarly. \( \diamond \)

For \( 0 < a, b < \infty \), let
\[ \tau_{a,b} = \inf\{t \geq 0 : |B_t| \notin (-a,b)\}. \]
Corollary.

\[ P[\mathcal{B}_{\tau, b} = b] = \frac{a}{a + b}, \quad E(\tau, b) = ab. \]

Proof. It is first shown that \( E(\tau_{\alpha, b}) < \infty \). Let \( \eta_n = \tau_{\alpha, b} \land n \). Then \( E(\eta_n) = E[\mathcal{B}_{\eta_n}^2] \leq \max(a^2, b^2) \) and, therefore, \( E(\tau_{\alpha, b}) = \lim_{n \to \infty} E(\eta_n) \leq \max(a^2, b^2) \). Next, let \( q = P[\mathcal{B}_{\tau, b} = b] \). Then \( P[\mathcal{B}_{\tau, b} = a] = 1 - q \),

\[ 0 = E(\mathcal{B}_{\tau, b}) = qb - (1 - q)a; \]

the first assertion follows from solving for \( q \); and the second follows from

\[ E(\tau_{\alpha, b}) = E(\mathcal{B}_{\tau, b}^2) = qb^2 + (1 - q)a^2 = ab. \]