Complex Integration
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Complex Numbers. Let $\mathbb{C}$ denote the complex numbers. Thus, each $z \in \mathbb{C}$ may be written $z = x + iy$, where $i^2 = -1$ and $x, y \in \mathbb{R}$ are called the real and imaginary parts of $z$. The reader is assumed to be familiar with complex numbers including addition, multiplication, and absolute value

$$z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2),$$
$$z_1z_2 = x_1x_2 - y_1y_2 + i(x_1y_2 + x_2y_1),$$
$$\bar{z} = x - iy,$$
$$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}.$$

If $\theta \in \mathbb{R}$, then the complex exponential is defined by

$$e^{i\theta} = \cos(\theta) + i\sin(\theta).$$

It is easily seen that $e^{i\theta} \times e^{i\omega} = e^{i(\theta + \omega)}$ and that $|e^{i\theta}| = 1$. Any complex number $z$ may be written as $z = re^{i\theta}$, where $r = |z|$ and $-\pi < \theta \leq \pi$; and if $z \neq 0$, then $\theta$ is uniquely determined.

Distance in $\mathbb{C}$ is defined by $d(z_1, z_2) = |z_1 - z_2|$. Viewed as a metric space, $\mathbb{C}$ is isometric to $\mathbb{R}^2$. The Borel sets of $\mathbb{C}$ is the smallest sigma-algebra containing the open sets.

Complex Integration. Now, let $(\Omega, \mathcal{A})$ be a measurable space. Then any function $f : \Omega \to \mathbb{C}$ may be written as $f = u + iv$, where $u(\omega)$ and $v(\omega)$ are the real and imaginary parts of $f(\omega)$ for each $\omega \in \Omega$. Clearly, $f$ is measurable iff $u$ and $v$ are measurable in which case $|f| = \sqrt{(u^2 + v^2)}$ is measurable. If $\mu$ is a measure on $\mathcal{A}$, then a measurable $f : \Omega \to \mathbb{C}$ is said to be integrable (with respect to $\mu$) iff $|f|$ is integrable in which case

$$\int_{\Omega} f d\mu = \int_{\Omega} u d\mu + i \int_{\Omega} v d\mu.$$

Proposition 1. If $f, f_1, f_2 : \Omega \to \mathbb{C}$ are integrable and $\alpha_1, \alpha_2 \in \mathbb{C}$, then

$$\int_{\Omega} (\alpha_1 f_1 + \alpha_2 f_2) d\mu = \alpha_1 \int_{\Omega} f_1 d\mu + \alpha_2 \int_{\Omega} f_2 d\mu$$

and

$$|\int_{\Omega} f d\mu| \leq \int_{\Omega} |f| d\mu.$$

Proof. The first assertion is left as an exercise, and the second is clear if $\int_{\Omega} f d\mu = 0$. If $\int_{\Omega} f d\mu \neq 0$, then $\int_{\Omega} f d\mu = re^{i\theta}$, where $r = |\int_{\Omega} f d\mu|$. In this case,

$$|\int_{\Omega} f d\mu| = r = \int_{\Omega} e^{-i\theta} f d\mu = \int_{\Omega} \Re[e^{-i\theta} f] d\mu \leq \int_{\Omega} |e^{-i\theta} f| d\mu \leq \int_{\Omega} |f| d\mu,$$

as asserted.✿
**Proposition 2.** The Dominated Convergence Theorem. Let \( f, f_1, f_2, \ldots \) be integrable complex valued functions. If \( \lim_{n \to \infty} f_n(\omega) = f(\omega) \) for a.e. \( \omega \), and if there is an integrable \( g \) for which \( |f_n| \leq g \) for all \( n \), then

\[
\lim_{n \to \infty} \int_{\Omega} f_n d\mu = \int_{\Omega} f d\mu.
\]

**Proof.** We have

\[
|\int_{\Omega} (f_n - f) d\mu| \leq \int_{\Omega} |f_n - f| d\mu
\]

for each \( n \). Here \( |f_n - f| \to 0 \) a.e. as \( n \to \infty \), and \( |f_n - f| \leq 2g \) for all \( n \). So,

\[
\lim_{n \to \infty} \int_{\Omega} |f_n - f| d\mu = 0
\]

by the Dominated Convergence Theorem for real valued functions. \( \diamond \)

**Remark.** If \( F : \mathbb{R} \to \mathbb{C} \) and the real and imaginary parts of \( F = U + iV \) are continuously differentiable, then \( F' = U' + iV' \) and

\[
\int_{(a,b]} F'(t) dt = \int_{a}^{b} F'(t) dt = F(b) - F(a).
\]

**Complex Random Variables.** If \((\Omega, \mathcal{A}, P)\) is a probability space, then a measurable function \( Z : \Omega \to \mathbb{C} \) is called a complex random variable. Write \( Z = X + iY \). A family \( Z_i, i \in I \), of complex random variables is said to be independent if \((X_i, Y_i), i \in I\), are.

**Proposition 3.** If \( Z_1, \ldots, Z_m \) are integrable, independent complex random variables, then

\[
E[Z_1 \times \cdots \times Z_m] = E(Z_1) \times \cdots \times E(Z_m).
\]

**Proof.** If \( m = 2 \), then

\[
E(Z_1Z_2) = E[(X_1X_2 - Y_1Y_2) + i(X_1Y_2 + X_2Y_1)]
\]

\[
= E(X_1)E(X_2) - E(Y_1)E(Y_2) + i[E(X_1)E(Y_2) + E(X_2)E(Y_1)]
\]

\[
= E(Z_1)E(Z_2).
\]

The general case then follows by induction. \( \diamond \)

**Characteristic Functions**

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**Fourier Transforms.** Now let \( F \) be a finite measure on the Borel sets of \( \mathbb{R} \). Then the **Fourier Transform of \( F \)** is defined by

\[
\hat{F}(t) = \int_{\mathbb{R}} e^{itx} F(dx) = \int_{\mathbb{R}} \cos(tx) F(dx) + i \int_{\mathbb{R}} \sin(tx) F(dx)
\]
for $t \in \mathbb{R}$. If $F$ is a probability measure and if $X$ is any random variable with distribution $F$, then

$$
\hat{F}(t) = E[e^{itX}]
$$

for $t \in \mathbb{R}$, by the Transformation Theorem. In this case $\hat{F}$ is called the characteristic function of $F$ or of $X$.

**Remark:** Linear Functions. If $X \sim F$ and $Y = aX + b \sim G$, then $\hat{G}(t) = e^{ibt} \hat{F}(at)$, since

$$
\hat{G}(t) = E[e^{itY}] = e^{ibt} E[e^{iatX}] = e^{ibt} \hat{F}(at).
$$

**Example 1.** If $F\{\pm 1\} = 1/2$, then $\hat{F}(t) = (e^{it} + e^{it})/2 = \cos(t)$.

**Example 2:** Exponential Distributions. If $F$ is the exponential distribution with failure rate $\lambda$, then

$$
\hat{F}(t) = \lim_{n \to \infty} \int_0^n \lambda e^{itx-\lambda x} dx = \lim_{n \to \infty} \frac{\lambda e^{itx-\lambda x}}{\lambda - it} \bigg|_{x=0}^{x=n} = \frac{\lambda}{\lambda - it}.
$$

If

$$
F\{dx\} = \frac{1}{2} \lambda e^{-\lambda|x|} dx, -\infty < x < \infty,
$$

then

$$
\hat{F}(t) = \frac{1}{2} \left[ \frac{\lambda}{\lambda - it} + \frac{\lambda}{\lambda + it} \right] = \frac{\lambda}{\lambda^2 + t^2}
$$

for $-\infty < t < \infty$. 

**Proposition 1.** Elementary Properties. If $F$ is a finite measure, then $|\hat{F}(t)| \leq \hat{F}(0) = F\{\mathbb{R}\}$ and $\hat{F}(-t) = \hat{F}(t)^\ast$ (complex conjugate) for all $t$ and $\hat{F}$ is uniformly continuous.

**Proof.** The first three assertions are clear. For the last,

$$
|\hat{F}(t + h) - \hat{F}(t)| = \left| \int_{\mathbb{R}} [e^{ithx} - 1] e^{itx} F\{dx\} \right| \leq \int_{\mathbb{R}} |e^{ithx} - 1| F\{dx\}
$$

for $t \in \mathbb{R}$ and $h > 0$. The right side does not depend on $t$ and approaches zero as $h \to 0$ by the Dominated Convergence Theorem.

In the next proposition, let $L_{c,h} = \{c \pm kh : k = 0, 1, 2, \ldots\}$ for $c \in \mathbb{R}$ and $h > 0$.

**Proposition 2.** If $F$ is a distribution function, then $F\{L_{c,h}\} = 1$ for some $c \in \mathbb{R}$ and $h > 0$ iff $|\hat{F}(t_0)| = 0$ for some $t_0 > 0$.

**Proof.** If $F\{L_{c,h}\} = 1$, where $h > 0$, let $t_0 = 2\pi/h$. Then $e^{it_0(x-c)} = 1$ a.e. $(F)$ and, therefore,

$$
e^{-it_0} \hat{F}(t_0) = \int_{\mathbb{R}} e^{it_0(x-c)} F\{dx\} = 1.
$$

Conversely, if $|\hat{F}(t_0)| = 1$ for some $t_0 > 0$, then $\hat{F}(t_0) = e^{it_0}$ for some $c \in \mathbb{R}$. Let $h = 2\pi/t_0$. Then

$$
0 = 1 - e^{-it_0} \hat{F}(t_0) = \int_{\mathbb{R}} [1 - e^{it_0(x-c)}] F\{dx\} = \int_{\mathbb{R}} [1 - \cos(t_0(x-c))] F\{dx\}
$$

and, therefore, $\cos(t_0(x-c)) = 1$ a.e. $(F)$. That is, $F\{L_{c,h}\} = 1$.

**Remark.** $F$ is said to be a lattice distribution of $F\{L_{c,h}\} = 1$ for some $c \in \mathbb{R}$ and $h > 0$. 

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Proposition 3: Convolutions. If $X \sim F$ and $Y \sim G$ be independent random variables, then $Z = X + Y$ has distribution function

$$K(z) = \int_R G(z - x)F\{dx\}.$$ 

If $G$ has a density $g$ (with respect to Lebesgue measure), then $K$ has density

$$k(z) = \int_R g(z - x)F\{dx\}.$$ 

Proof. The joint distribution of $X$ and $Y$ is $F \times G$, the product measure. Fix and $z$ and let $B = \{(x, y) : x + y \leq z\}$. Then $P\{Z \leq z\} = (F \times G)(B)$. So, by Fubini’s Theorem,

$$K(z) = \int_R G\{B_x\}F\{dx\} = \int_R G(z - x)F\{dx\},$$

as asserted. If $G$ has density $g$, then the right side is

$$\int_R \left[ \int_{(-\infty, z-x]} g(y)dy \right] F\{dx\} = \int_R \left[ \int_{(-\infty, z]} g(y-x)dy \right] F\{dx\} = \int_{(-\infty, z]} k(y)dy,$$

where $k$ is as in the statement of the proposition, by a change of variables and Fubini’s Theorem. So, $K$ has density $k$. ◇

Example 3. If $F$ and $G$ are uniform on $(-\frac{1}{2}, \frac{1}{2})$, then $k(z) = (1 - |z|)^+$. ◇

Remark. $K$ is called the convolution of $F$ and $G$ and denoted by $K = F * G$. It follows directly from the proposition that $F * G = G * F$ and $(F * G) * H = F * (G * H)$ for distribution functions $F$, $G$, and $H$. Thus "F * G * H" is unambiguous. ◇

Proposition 4. More on Convolutions. If $F_1, \cdots, F_n$ are probability measures, then $(F_1 * \cdots * F_n)^r(t) = \hat{F}_1(t) \times \hat{F}_n(t)$ for all $t$.

Proof. Construct independent random variables $X_1, \cdots, X_n$ with distribution functions $F_1, \cdots, F_n$. Then $X_1 + \cdots + X_n$ has distribution function $F_1 * \cdots * F_n$ and, therefore,

$$(F_1 * \cdots * F_n)^r(t) = E[e^{it(X_1 + \cdots + X_n)}]$$

$$= E[e^{itX_1}] \times \cdots \times E[e^{itX_n}]$$

$$= \hat{F}_1(t) \times \cdots \times \hat{F}_n(t)$$

for all $t$. ◇

Example 1: Revisited. If $X_1, \cdots, X_n \sim^{ind} F$, where $F\{\pm 1\} = 1/2$, then the characteristic function of $S_n = X_1 + \cdots + X_n$ is $\cos^n(t)$. 4
Moments and Derivatives. Recall that the moments of a distribution $F$ are $\mu_k = \int_{\mathbb{R}} x^k F\{dx\}$, if these exist. The same notation is used when $F$ is a finite measure. There is a simple relation between the moments of a distribution and derivatives of the characteristic function. Let

$$\rho_k(t) = e^{it} - \sum_{j=0}^{k-1} \frac{1}{j!} (it)^j$$

**Lemma 1.**

$$|\rho_k(t)| \leq \frac{1}{k!} |t|^k.$$

**Proof.** If $k = 1$, then

$$\rho_1(t) = |e^{it} - 1| = \left| \int_0^t e^{is} ds \right| \leq \int_0^{|t|} ds = |t|,$$

Suppose that the Lemma were known for $k < m$, where $m = ge2$. Then $\rho'_m(t) = \rho_{m-1}(t)$, so that

$$|\rho_m(t)| = \left| \int_0^t \rho_{m-1}(s) ds \right| \leq \int_0^{|t|} \frac{s^{m-1}}{(m-1)!} ds = \frac{|t|^m}{m!},$$

as asserted. \hfill \Box

**The Moments Theorem.** If $k \geq 1$ and $F$ has a finite $k^{th}$ moment, then then $\hat{F}$ has $k$ continuous derivative given by

$$\hat{F}^j(t) = i^j \int_{\mathbb{R}} x^j e^{itx} F\{dx\} \quad (!)$$

for $t \in \mathbb{R}$ and $j = 1, \ldots, k$. Conversely, if $\hat{F}''(0)$ exist (finite), then $F$ has a finite second moment.

**Proof.** Suppose first that $j = 1$. If $h_n \to 0$, $h_n \neq 0$, and $t \in \mathbb{R}$, then

$$\frac{\hat{F}(t+h_n) - \hat{F}(t)}{h_n} = \int_{\mathbb{R}} g_n(x) F\{dx\},$$

where

$$g_n(x) = \frac{e^{ih_n x} - 1}{h_n} e^{itx}$$

for $x \in \mathbb{R}$. Clearly, $\lim_{n \to \infty} g_n(x) = ix e^{tx}$ for all $x$, and $|g_n(x)| \leq |x|$ for all $n$ and $x$. So,

$$\lim_{n \to \infty} \int_{\mathbb{R}} g_n(x) F\{dx\} = \int_{\mathbb{R}} ix e^{tx} F\{dx\},$$

by the Dominated Convergence Theorem. It follows that $\hat{F}$ is differentiable and that $\hat{F}'$ is given by (!). That $\hat{F}'$ is uniformly continuous, then follows from Proposition 1, since

$$\hat{F}'(t) = i \int_{[0,\infty)} e^{itx} x F\{dx\} - i \int_{[0,\infty)} e^{itx} |x| F\{dx\},$$

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and both integrals are the Fourier transforms of finite measures. The general case \( j \geq 2 \) now follows by induction.

For the converse, if \( F \) is continuously differentiable on some neighborhood of 0 and \( F''(0) \) exists (finite), then

\[
-\hat{F}''(0) = \lim_{t \downarrow 0} \frac{2 - \hat{F}(t) - \hat{F}(-t)}{t^2}.
\]

The right side is

\[
\lim_{t \downarrow 0} 2 \int_{\mathbb{R}} \frac{1 - \cos(tx)}{t^2} F(dx) \geq \int_{\mathbb{R}} x^2 F(dx),
\]

by Fatou's Lemma.

\[\diamondsuit\]

**Corollary.** If \( \mu_k \) is finite, then \( \hat{F}^{(k)}(0) = i^k \mu_k \).

**Normal Distributions.** Recall that the standard normal distribution function is

\[
\Phi(z) = \int_{(-\infty, z]} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy.
\]

The normal distribution function with parameter \( \mu \in \mathbb{R} \) and \( \sigma^2 > 0 \) is

\[
F(x) = \Phi\left( \frac{x - \mu}{\sigma} \right).
\]

**Proposition 5.** If \( F \) is normal with parameters \( \mu \) and \( \sigma^2 \), then

\[
\hat{F}(t) = e^{i\mu t - \frac{1}{2}\sigma^2 t^2}
\]

for \( t \in \mathbb{R} \).

**Proof.** Suppose first that \( \mu = 0 \) and \( \sigma^2 = 1 \), so that \( F = \Phi \). Then

\[
\hat{\Phi}''(t) = -\int_{-\infty}^{\infty} x^2 e^{itx} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \cos(tx) e^{-\frac{1}{2}x^2} dx,
\]

by Moments Theorem. Integrating the right side by parts twice, then leads to

\[
\Phi''(t) = \frac{t^2 - 1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(tx) e^{-\frac{1}{2}x^2} = (t^2 - 1)\Phi(t)
\]

for \( t \in \mathbb{R} \). Moreover, \( \Phi(0) = 1 \) and \( \Phi'(0) = 0 \). The unique solution this differential equation is \( \hat{\Phi}(t) = e^{-\frac{1}{2}t^2} \), as asserted.

For the general case, let \( Z \sim \Phi \) and \( X = \sigma Z + \mu \). Then \( X \sim F \) and, therefore, \( \hat{F}(t) = e^{i\mu t} \Phi(\sigma t) \).

\[\diamondsuit\]

**Corollary.** The mean and variance of \( F \) are \( \mu \) and \( \sigma^2 \)

**The Riemann Lebesgue Lemma.** If \( dF = f d\lambda \), where \( f \geq 0 \) is integrable, then \( \lim_{|t| \to \infty} \hat{F}(t) = 0 \).

**Proof.** See text.