Instructions: Do any 5 problems. Show your work and explain your reasoning carefully. Additional options may be added in class.

1. Let $Y$ and $Z$ be independent random variables for which $Y$ has the standard exponential distribution and $Z$ has the standard normal distribution. Find the distribution functions and characteristic function of $X = \sqrt{2Y} \times Z$.

2. A probability distribution $F$ is said to be arithmetic if there is an $h > 0$ for which

$$F\{\cdots - h, 0, h, 2h, \ldots\} = 1.$$  \hspace{1cm} (1)

a) Show that $F$ is arithmetic iff $\hat{F}$ is periodic.

b) Show that if $F$ is arithmetic and $F\{0\} < 1$, then there is a largest $h > 0$ for which (1) holds.

Notes: If $\hat{F}$ is periodic, then there is a $t_0 > 0$ for which $\hat{F}(t_0) = 1$. The $h$ found in b) is called the arithmetic span of $F$.

3. Let $F$ be a distribution function; write $\Delta F(x) = F(x) - F(x-)$. and let $D_F$ be the discontinuity set of $F$, $D_F = \{x \in \mathbb{R} : \Delta F(x) > 0\}$. Show that

$$\Delta F(x) = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} e^{-ix} \hat{F}(t) dt$$  \hspace{1cm} (2)

for all $x \in \mathbb{R}$ and that

$$\lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\hat{F}(t)|^2 dt = \sum_{x \in D_F} \Delta F(x)^2.$$  

Simplify the right side of (2) when $F$ is arithmetic.

4. a). Show that if $0 < \alpha < 2$, then

$$\kappa_\alpha = \lim_{y \to \infty} \int_{0}^{y} \frac{\sin(x)}{x^\alpha} dx$$

exists and is finite.

b). Now let $F$ be a symmetric probability distribution for which $\lim_{x \to \infty} x^\alpha [1 - F(x)] = c$ where $0 < \alpha < 2$ and $0 < c < \infty$. Show that

$$\lim_{t \downarrow 0} \frac{1 - \hat{F}(t)}{t^\alpha} = 2\kappa_\alpha c.$$  

Notes: The problem is simpler when $1 < \alpha < 2$; Fubini’s Theorem and the Dominated Convergence Theorem are useful tools for b).
5. Let $X_{n1}, \ldots, X_{nn}$ be a triangular array of independent random variables with means $E(X_{nk}) = 0$ and finite variances $\sigma^2_{nk} = E(X^2_{nk})$ for which $\sigma^2_{n1} + \cdots + \sigma^2_{nn}$ is bounded in $n$. Show that if the Lindeberg-Feller Condition is satisfied, then

$$\max_{k \leq n} |X_{nk}| \to 0$$

in probability as $n \to \infty$.

**Hint:** Use Boole's Inequality.

6. Let $X_{n1}, \ldots, X_{nn}$ be a uniformly asymptotically negligible triangular array (of independent random variables) for which $0 \leq X_{nk} \leq 1$ w.p.1 for all $n$ and $k$. Let $F_{nk}$ be the distribution function of $X_{nk}$ and

$$H_n(x) = \sum_{k=1}^{n} [1 - F_{nk}(x)].$$

Suppose that

$$H(x) = \lim_{n \to \infty} H_n(x)$$

exists and is finite for all $0 < x \leq 1$ and that

$$\lim_{n \to \infty} \int_{[0,1]} H_n(x) dx = \int_{[0,1]} H(x) dx.$$

Find the limit of the characteristic function of $S_n = X_{n1} + \cdots + X_{nn}$.

**Hint:** Use an extended version of the Dominated Convergence Theorem.

7. Let $F$ be a distribution function with mean $\mu = 0$ and variance $\sigma^2 = 1$, and suppose that $F$ satisfies Cramer’s Condition,

$$\limsup_{t \to \infty} |\hat{F}(t)| < 1.$$

Let $X_1, X_2, \cdots \sim ind$ $F$ be i.i.d. and let $S_n = X_1 + \cdots + X_n$ for $n \geq 1$. Next, let $g$ be a bounded, continuous, integrable function and let

$$\tilde{g}(t) = \int_{\mathbb{R}} e^{itx} g(x) dx$$

for $t \in \mathbb{R}$. Show that

$$\lim_{n \to \infty} \sqrt{n} E[\tilde{g}(S_n)] = \sqrt{2\pi} g(0).$$

**Hint:** Use Parseval’s Relation (applied to the positive and negative parts of $g$).

8. Let $p_1, \ldots, p_m$ be positive probabilities, so that $p_i > 0$ for all $i$ and $p_1 + \cdots + p_m = 1$. Let $Y_1, Y_2, \cdots$ be independent random variables for which $P(Y_j = k) = p_k$ for all $j$ and $k$, and let

$$N_{n,k} = \#\{j \leq n : Y_j = k\}, \ k = 1, \cdots, m,$$

and

$$W_n = \sum_{k=1}^{m} \frac{(N_{n,k} - np_k)^2}{np_k}.$$ 

Show that the distribution function of $W_n$ converges weakly to the chi-squared distribution on $m - 1$ degrees of freedom. (The latter is defined in Problem 20.22 of the text.)

**Hint:** Represent $W_n$ as a continuous function of a random vector $Z_n$ that has a limiting normal distribution.