Signed Measures and The Radon Nikodym Theorem
February 7, 2001

Signed Measures. Let \((\Omega, \mathcal{A})\) be a measurable space. Then a finite signed measure is a function
\[
\psi : \mathcal{A} \to \mathbb{R}
\]
for which
\[
\psi\left( \bigcup_{k=1}^{\infty} A_k \right) = \sum_{k=1}^{\infty} \psi(A_k)
\]
whenever \(A_1, A_2, \cdots \in \mathcal{A}\) are mutually exclusive. Thus, infinite values are not allowed for \(\psi\). Observe that
\[
\psi(\emptyset) = 0
\]
for any finite signed measure \(\psi\).

Example 1. Any finite measure is a finite signed measure. If \(\mu\) and \(\nu\) are finite signed measures, then
\[
\psi(A) = \nu(A) - \mu(A), \quad A \in \mathcal{A},
\]
defines a finite signed measure. \(\diamond\)

Example 2. If \(\mu\) and any measure and \(f : \Omega \to \mathbb{R}\) is integrable with respect to \(\mu\), then
\[
\psi(A) = \int_A f d\mu, \quad A \in \mathcal{A},
\]
defines a finite signed measure, by an easy application of the Dominated Convergence Theorem. \(\diamond\)

Proposition. The Monotone Sequences Theorem. Let \(\psi\) be a signed measure. If \(A_n \in \mathcal{A}\) and \(A_n \uparrow A\) or \(A_n \downarrow A\) as \(n \to \infty\), then
\[
\lim_{n \to \infty} \psi(A_n) = \psi(A).
\]

Proof. Exercise—similar to the case of measures.

If \(\psi\) is a finite signed measure, then a set \(B \in \mathcal{A}\) is called a positive set for \(\psi\) if \(\psi(A) \geq 0\) for all \(A \in \mathcal{A} \cap B\), and a negative set for \(\psi\) if \(\psi(A) \leq 0\) for all \(A \in \mathcal{A} \cap B\).

Example 2: Continued. If \(\psi\) is as in (1), then \(B = \{\omega : f(\omega) \geq 0\}\) is a positive set for \(\psi\). \(\diamond\)

The Hahn Decomposition Theorem. If \(\psi\) is any finite signed measure, then there are a positive set \(\Omega^+\) and a negative set \(\Omega^-\) for which \(\Omega^+ \cap \Omega^- = \emptyset\) and \(\Omega = \Omega^+ \cup \Omega^-\).

Proof. The set \(\Omega^+\) will be so chosen that \(\psi(\Omega^+) = \sup_{A \in \mathcal{A}} \psi(A) = \alpha\), say. There are \(A_n \in \mathcal{A}\) for which
\[
\lim_{n \to \infty} \psi(A_n) = \alpha = \sup_{A \in \mathcal{A}} \psi(A).
\]
Let
\[ C_n = \{B_1 \cap \cdots \cap B_n : B_j = A_j \text{ or } A'_j\}, \]
\[ C_n^o = \{C \in C_n : B_n = A_n\}, \]
\[ C_n^+ = \{C \in C_n^o : \psi(C) > 0\}, \]
and
\[ E_n = \bigcup_{C \in C_n^+} C. \]
Then
\[ \psi(A_n) = \sum_{C \in C_n^o} \psi(C) \leq \sum_{C \in C_n^+} \psi(C) = \psi(E_n) \]
for all \( n \geq 1 \). Clearly,
\[ E_m \cup \cdots \cup E_n \subseteq E_m \cup \cdots \cup E_{n+1} \]
and
\[ \psi(E_m \cup \cdots \cup E_n) \leq \psi(E_m \cup \cdots \cup E_{n+1}), \]
and for \( 0 \leq n < \infty \), because we add only terms with positive measure at each stage. Let
\[ F_m = \bigcup_{n=m}^{\infty} E_n. \]
Then
\[ \psi(A_m) \leq \psi(E_m) \leq \lim_{n \to \infty} \psi(E_m \cup \cdots \cup E_n) = \psi(F_m) \]
for all \( m \). Now let
\[ \Omega^+ = \bigcap_{m=1}^{\infty} F_m. \]
Then
\[ \alpha = \lim_{m \to \infty} \psi(A_m) \leq \lim_{m \to \infty} \psi(F_m) = \psi(\Omega^+) \]
and, therefore,
\[ \alpha = \psi(\Omega^+) < \infty. \]
Let \( \Omega^- = \Omega - \Omega^+ \). Then, it is clear that \( \Omega^+ \) is a positive set. For if \( B \subset A \cap \Omega^+ \) and \( \psi(B) < 0 \), then
\[ \psi(\Omega^+) = \psi(B) + \psi(\Omega^+ - B) < \psi(\Omega^+ - B), \]
contradicting \( \psi(\Omega^+) = \alpha \). That \( \Omega^- \) is a negative set may be established similarly. \( \diamond \)
Corollary 1.

\[
\psi(\Omega^+) = \sup_{A \in \mathcal{A}} \psi(A), \\
\psi(\Omega^-) = \sup_{A \in \mathcal{A}} \psi(A),
\]

Corollary 2. The Jordan Decomposition. For \( B \in \mathcal{A} \), let

\[
\psi^+(B) = \psi(\Omega^+ \cap B), \\
\psi^-(B) = -\psi(\Omega^- \cap B).
\]

Then \( \psi^\pm \) are finite measures,

\[
\psi(B) = \psi^+(B) - \psi^-(B), \\
\psi^+(B) = \sup_{A \subseteq B} \psi(A), \\
\psi^-(B) = -\inf_{A \subseteq B} \psi(A)
\]

for \( B \in \mathcal{A} \).

Proof. The first three assertions are clear. For the fourth observe that if \( A \subseteq B \), then \( \psi(A) = \psi^+(A) - \psi^-(A) \leq \psi^+(B) \) with equality if \( A = \Omega^+ \cap B \). The fifth assertion may be established similarly. \( \diamond \)

Remark. The sets \( \Omega^\pm \) are not uniquely determined, but the measures \( \psi^\pm \) are.

Singularity and Absolute Continuity. Two measures \( \mu \) and \( \nu \) are said to be mutually singular if there is a measurable \( \Omega_0 \) for which

\[
\mu(\Omega - \Omega_0) = 0 \quad \text{and} \quad \nu(\Omega_0) = 0.
\]

This relationship is denoted by \( \mu \perp \nu \). For example, the measures \( \psi^+ \) and \( \psi^- \) of the last corollary are mutually singular.

If \( \mu \) and \( \nu \) are measures on \( \mathcal{A} \), then \( \nu \) is said to be absolutely continuous with respect to \( \mu \) if \( \nu(A) = 0 \) whenever \( A \in \mathcal{A} \) and \( \mu(A) = 0 \). This relation is denoted by \( \nu \ll \mu \).

Example 3. If \( \mu \) is a measure and \( f \) is a non-negative measurable function, then

\[
\nu(A) = \int_A f \, d\mu, \quad A \in \mathcal{A},
\]

defines a measure which is absolutely continuous with respect to \( \mu \).

The Radon Nikodym Theorem provides a converse to the example.
Lemma 1. If \( \nu \) is finite and \( \nu \ll \mu \) then: \( \forall \epsilon > 0, \exists \delta > 0 \) for which \( \nu(A) \leq \epsilon \) whenever \( A \in \mathcal{A} \) and \( \mu(A) \leq \delta \).

Proof. If the condition were not satisfied, then there would be an \( \epsilon > 0 \) and a sequence of sets \( A_n \in \mathcal{A} \) for which \( \mu(A_n) \leq \frac{1}{n^2} \) and \( \nu(A_n) \geq \epsilon \) for all \( n \geq 1 \). Let \( B_n = A_n \cup A_{n+1} \cup \cdots \) and \( C = B_1 \cap B_2 \cap \cdots \). Then

\[
\nu(C) = \lim_{n \to \infty} \nu(B_n) \geq \liminf_{n \to \infty} \nu(A_n) \geq \epsilon
\]

and

\[
\mu(C) = \lim_{n \to \infty} \mu(B_n) \leq \limsup_{n \to \infty} \sum_{k=n}^{\infty} \mu(A_k) = 0,
\]

contradicting the assumed absolute continuity.

\( \diamond \)

Remark. If \( \nu \) is finite, then the condition is, in fact, necessary and sufficient for absolute continuity.

Example 4. Let \( \Omega = \{1, 2, \cdots\} \), \( \mathcal{A} = \mathcal{P}(\Omega) \), \( \nu = \) counting measure, and

\[
\mu(A) = \sum_{n \in A} \frac{1}{n^2}.
\]

Then

\[
\nu \ll \mu,
\]

\[
\lim_{n \to \infty} \mu\{n + 1, n + 2, \cdots\} = 0,
\]

but

\[
\nu\{n + 1, n + 2, \cdots\} = \infty
\]

for all \( n \).

\( \diamond \)

Lemma 2. Let \( \mu \) and \( \nu \) be finite measures that are not mutually singular. Then there is an \( \epsilon > 0 \) and an \( B \in \mathcal{A} \) for which \( \mu(B) > 0 \) and \( \nu(A) \geq \epsilon \mu(A) \) for all \( A \in \mathcal{A} \cap B \).

Proof. Let

\[
\psi_n(A) = \nu(A) - \frac{1}{n} \mu(A)
\]

for \( A \in \mathcal{A} \). Then each \( \psi_n \) is a finite signed measure. Let \( B^+_n \) and \( B^-_n \) be a Hahn Decomposition of \( \Omega \) for \( \psi_n \). Further, let

\[
C = \bigcup_{n=1}^{\infty} B^+_n.
\]

Then

\[
C' = \bigcap_{n=1}^{\infty} B^-_n.
\]

Then \( \psi_n(C') \leq 0 \) for all \( n \), since \( C' \subset B^-_n \) for all \( n \). It follows that \( \nu(C') \leq \mu(C')/n \) for all \( n \) and, therefore, that \( \nu(C') = 0 \). Since \( \mu \) and \( \nu \) are not mutually singular, it then follows that \( \mu(C) > 0 \). So, \( \mu(B^+_m) > 0 \) for some \( m \), and the lemma then follows with \( B = B^+_m \) and \( \epsilon = 1/m \). For if \( A \subseteq B^+_m \), then \( \psi_m(A) \geq 0 \) and, therefore, \( \nu(A) \geq \mu(A)/m \).

\( \diamond \)
Lemma 3. Let \( \mu \) and \( \nu \) be measures; and let \( \mathcal{G} \) be the collection of all non-negative measurable \( g : \Omega \to \mathbb{R} \) for which \( \int_A g d\mu \leq \nu(A) \) for all \( A \in \mathcal{A} \). Then \( \mathcal{G} \) is closed under the formation of maxima and increasing limits.

**Proof.** If \( g_1, g_2 \in \mathcal{G} \), then \( g = g_1 \vee g_2 \in \mathcal{G} \) because

\[
\int_A g d\mu = \int_{A, g_1 \leq g_2} g_2 d\mu + \int_{A, g_2 < g_1} g_1 d\mu \\
\leq \nu(A \cap \{g_1 \leq g_2\}) + \nu(A \cap \{g_2 < g_1\}) \\
= \nu(A)
\]

for all \( A \in \mathcal{A} \). Similarly, if \( g_n \in \mathcal{G} \) and \( g_n \uparrow g \), then \( g \in \mathcal{G} \) because

\[
\int_A g d\mu = \lim_{n \to \infty} \int_A g_n d\mu \leq \nu(A)
\]

for all \( A \in \mathcal{A} \). \( \diamond \)

Lemma 4. Suppose that \( \mu \) is sigma-finite. If \( f \) and \( g \) are non-negative, measurable functions for which

\[
\int_A g d\mu \leq (=) \int_A f d\mu
\]

for all \( A \in \mathcal{A} \), then \( g \leq (=) f \text{ a.e. } (\mu) \).

**Proof.** There are \( \Omega_m \in \mathcal{A} \) for which \( \mu(\Omega_m) < \infty \) for all \( m \) and \( \Omega_m \uparrow \Omega \) as \( n \to \infty \). Let \( \Omega_{mn} = \{\omega : f(\omega) \leq n\} \cap \Omega_m \). Then \( \int_{\Omega_{mn}} g d\mu = \int_{\Omega_{mn}} f d\mu < \infty \) for all \( m \) and \( n \). Let \( B = \{\omega : g(\omega) > f(\omega)\} \). Then

\[
\int_{\Omega_{mn}} (g - f)_+ d\mu = \int_{B \cap \Omega_{mn}} (g - f)d\mu = 0
\]

for all \( m \) and \( n \). So, \( \mu(\{\omega : g(\omega) > f(\omega)\} \cap \Omega_{mn}) = 0 \) for all \( m \) and \( n \). It follows easily that \( g \leq f \text{ a.e.} \). If there is equality in (2), then \( f \leq g \text{ a.e.} \), by reversing the roles of \( f \) and \( g \), and \( f = g \text{ a.e.} \). \( \diamond \)

The Radon Nikodym Theorem. If \( \mu \) and \( \nu \) are sigma-finite measures for which \( \nu \ll \mu \), then there is a non-negative measurable \( f \) for which

\[
\nu(A) = \int_A f d\mu
\]

for all \( A \in \mathcal{A} \).

**Proof.** Suppose first that \( \mu \) and \( \nu \) are finite. Define \( \mathcal{G} \) as in the previous lemma. Then there are \( g_1, g_2, \cdots \in \mathcal{G} \) for which

\[
\lim_{n \to \infty} \int_{\Omega} g_n d\mu = \sup_{g \in \mathcal{G}} \int_{\Omega} g d\mu;
\]


and the right side is finite, because $\nu(\Omega) < \infty$. Let $f_n = g_1 \vee \cdots \vee g_n$. Then $f_n \uparrow f$ and $f \in \mathcal{G}$ by the previous lemma. Let

$$\psi(A) = \nu(A) - \int_A f \, d\mu$$

for $A \in \mathcal{A}$. Then $\psi$ is a finite measure and $\psi \ll \mu$. So, if $\psi \neq 0$, then $\psi$ and $\mu$ cannot be singular and there are $B \in \mathcal{A}$ and $\epsilon > 0$ for which $\mu(B) > 0$ and $\psi(A) \geq \epsilon \mu(A)$ for all $A \in \mathcal{A} \cap B$. Let $h = f + 1_B$. Then

$$\int_A h \, d\mu = \int_A f \, d\mu + \epsilon \mu(A \cap B)$$

$$\leq \int_A f \, d\mu + \psi(A \cap B)$$

$$= \nu(A) - \psi(A) + \psi(A \cap B)$$

$$\leq \nu(A)$$

for all $A \in \mathcal{A}$. So, $h \in \mathcal{G}$; but $\int_\Omega h \, d\mu = \int_\Omega f \, d\mu + \epsilon \mu(B) > \int_\Omega f \, d\mu$, contradicting the construction of $f$. So, $\psi = 0$ and

$$\nu(A) = \int_A f \, d\mu$$

for all $A \in \mathcal{A}$.

For the sigma-finite case, we may write $\Omega = \bigcup_{m,n=1}^\infty \Omega_{mn}$, where $\Omega_{mn}$ are mutually exclusive sets for which $\mu(\Omega_{mn}) + \mu(\Omega_{mn}) < \infty$ for all $m$ and $n$. Let $\mu_{mn} = \mu|\Omega_{mn}$, the restriction of $\mu$ to $\Omega_{mn}$ and $\nu_{mn} = \mu|\Omega_{mn}$. Then it is easily seen that $\nu_{mn} \ll \mu_{mn}$ for all $m$ and $n$. So, there are measurable $f_{mn}$ for which

$$\nu_{mn}(A) = \int_A f_{mn} \, d\mu_{mn} = \int_A f_{mn} \, d\mu$$

for $A \in \mathcal{A} \cap \Omega_{mn}$. Let

$$f = \sum_{m,n=1}^\infty f_{mn} 1_{\Omega_{mn}}.$$ If $A \in \mathcal{A}$, then

$$\nu(A) = \sum_{m,n=1}^\infty \nu(A \cap \Omega_{mn}) = \sum_{m,n=1}^\infty \int_{A \cap \Omega_{mn}} f_{mn} \, d\mu$$

$$= \sum_{m,n=1}^\infty \int_{A \cap \Omega_{mn}} f \, d\mu = \int_A f \, d\mu,$$

as required.

The essential uniqueness follows from Lemma 4.

Any function $f$ for which (3) holds is called a version of the Radon Nikodym derivative of $\nu$ with respect to $\mu$ and denoted by

$$f = \frac{d\nu}{d\mu}.$$

Remark. The absolute continuity was not used in the construction of $f$. 

6
The Lebesgue Decomposition Theorem. If $\mu$ and $\nu$ are sigma-finite measures, then there are measures $\nu_a$ and $\nu_s$ for which $\nu = \nu_a + \nu_s$, $\nu_a \ll \mu$, and $\nu_s \perp \mu$.

Proof. Suppose first that $\mu$ and $\nu$ are finite, and construct $f$ and $\psi$, as in the proof of the Radon Nikodym Theorem.