Characteristic Functions
February 11 and 13, 2003

Notation: Write elements of $\mathbb{R}^p$ as column vectors $x = (x_1, \cdots, x_p)'$. Thus, $x'y = x_1y_1 + \cdots + x_py_p$. Also, write $|x|$ for the Euclidean norm $|x| = \sqrt{x'x}$.

Fourier Transforms. Now let $F$ be a finite measure on the Borel sets of $\mathbb{R}$. Then the Fourier Transform of $F$ is defined by

$$\hat{F}(t) = \int_{\mathbb{R}^p} e^{it'x}F(dx) = \int_{\mathbb{R}^p} \cos(t'x)F(dx) + i \int_{\mathbb{R}^p} \sin(t'x)F(dx)$$

for $t \in \mathbb{R}^p$. If $F$ is a probability measure and if $X$ is any random variable with distribution $F$, then

$$\hat{F}(t) = E[e^{itX}]$$

for $t \in \mathbb{R}$, by the Transformation Theorem. In this case $\hat{F}$ is called the characteristic function of $F$ or of $X$.

Example 1. If $p = 1$ and $F\{\pm 1\} = 1/2$, then $\hat{F}(t) = (e^{it} + e^{it})/2 = \cos(t)$.

Example 2: Exponential Distributions. If $p = 1$ and $F$ is the exponential distribution with failure rate $\lambda$, then

$$\hat{F}(t) = \lim_{n \to \infty} \int_{0}^{n} \lambda e^{itx-\lambda x} dx = \lim_{n \to \infty} \frac{\lambda e^{itx-\lambda x}}{\lambda - it} |_{x=0}^{n} = \frac{\lambda}{\lambda - it}.$$ 

If

$$F(dx) = \frac{1}{\lambda} e^{-\lambda|x|} dx, -\infty < x < \infty,$$

then

$$\hat{F}(t) = \frac{1}{2} \left[ \frac{\lambda}{\lambda - it} + \frac{\lambda}{\lambda + it} \right] = \frac{\lambda}{\lambda^2 + t^2}$$

for $-\infty < t < \infty$.

Proposition 1. Elementary Properties. If $F$ is a finite measure, then $|\hat{F}(t)| \leq \hat{F}(0) = F\{\mathbb{R}\}$ and $\hat{F}(-t) = \hat{F}(t)^*$ (complex conjugate) for all $t$; and $\hat{F}$ is uniformly continuous.

Proof. The first three assertions are clear. For the last,

$$|\hat{F}(t + h) - \hat{F}(t)| = \left| \int_{\mathbb{R}^p} [e^{ih'x} - 1] e^{it'x} F(dx) \right| \leq \int_{\mathbb{R}^p} |e^{ih'x} - 1| |F(dx)|$$

for $t \in \mathbb{R}$ and $h > 0$. The right side does not depend on $t$ and approaches zero as $h \to 0$ by the Dominated Convergence Theorem.

Proposition 2. Linear Functions. If $X \sim F$ and $Y = AX + b \sim G$, where $A$ is $q \times p$ and $b \in \mathbb{R}^q$ then $\hat{G}(t) = e^{ib't} \hat{F}(A't)$.

Proof. In fact,

$$\hat{G}(t) = E[e^{itY}] = e^{it'b} E[e^{it'AX}] = e^{ib't} \hat{F}(A't),$$

as asserted.
Proposition 3. Let $X_i \sim F_i$, $i = 1, \ldots, p$, be independent random variables; and let $F$ be the distribution of $(X_1, \ldots, X_p)'$. Then

$$\hat{F}(t) = \hat{F}_1(t_1) \times \cdots \times \hat{F}_p(t_p)$$

for $t \in \mathbb{R}^p$.

Proof. In this case,

$$\hat{F}(t) = E[e^{t_1 X_1 + \cdots + t_p X_p}] = E\left[ \prod_{j=1}^{p} e^{t_j X_j} \right] = \prod_{j=1}^{p} E\left[ e^{t_j X_j} \right] = \prod_{j=1}^{p} F_j(t_j)$$

for $t \in \mathbb{R}^p$. \hfill \diamond

In the next proposition, let $p = 1$ and $L_{c,h} = \{ c \pm kh : k = 0, 1, 2, \ldots \}$ for $c \in \mathbb{R}$ and $h > 0$.

Proposition 4. If $F$ is a distribution function, then $F\{L_{c,h}\} = 1$ for some $c \in \mathbb{R}$ and $h > 0$ iff $|\hat{F}(t_0)| = 0$ for some $t_0 > 0$.

Proof. If $F\{L_{c,h}\} = 1$, where $h > 0$, let $t_0 = 2\pi/h$. Then $e^{i t_0 (x-c)} = 1$ a.e. ($F$) and, therefore,

$$e^{-ict_0} \hat{F}(t_0) = \int_{\mathbb{R}} e^{it_0 (x-c)} F\{dx\} = 1.$$ 

Conversely, if $|\hat{F}(t_0)| = 1$ for some $t_0 > 0$, then $\hat{F}(t_0) = e^{ict_0}$ for some $c \in \mathbb{R}$. Let $h = 2\pi/t_0$. Then

$$0 = 1 - e^{-ict_0} \hat{F}(t_0) = \int_{\mathbb{R}} [1 - e^{it_0 (x-c)}] F\{dx\} = \int_{\mathbb{R}} [1 - \cos[t_0 (x-c)] F\{dx\}$$

and, therefore, $\cos[t_0 (x-c)] = 1$ a.e. ($F$). That is, $F\{L_{c,h}\} = 1$. \hfill \diamond

Remark. $F$ is said to be a lattice distribution if $F\{L_{c,h}\} = 1$ for some $c \in \mathbb{R}$ and $h > 0$. For example, if $F\{\pm 1\} = 1/2$, then $\hat{F}(t) = \cos(t)$.

Proposition 5: Convolutions. If $X \sim F$ and $Y \sim G$ be independent random $p$-vectors, then $Z = X + Y$ has distribution

$$K\{B\} = \int_{\mathbb{R}^p} G\{B-x\} F\{dx\},$$

for Borel sets $B \subseteq \mathbb{R}^p$, where $B - x = \{ y \in \mathbb{R}^p : x + y \in B \}$. If $G$ has a density $g$ (with respect to Lebesgue measure), then $K$ has density

$$k(z) = \int_{\mathbb{R}^p} g(z-x) F\{dx\}.$$
Proof. The joint distribution of $X$ and $Y$ is $F \times G$, the product measure. Fix and $z$ and let $C = \{(x, y) : x+y \leq z\}$. Then $P\{Z \in B\} = (F \times G)(C)$. So, by Fubini's Theorem,

$$K\{B\} = \int_{\mathbb{R}^p} G\{C_x\}F\{dx\} = \int_{\mathbb{R}^p} G\{B-x\}F\{dx\},$$

where $C_x = \{y : (x, y) \in C\}$. If $G$ has density $g$, then the right side is

$$\int_{\mathbb{R}^p} \left[ \int_{B-x} g(y)dy \right] F\{dx\} = \int_{\mathbb{R}^p} \left[ \int_B g(y-x)dy \right] F\{dx\} = \int_B k(y)dy,$$

where $k$ is as in the statement of the proposition, by a change of variables and Fubini's Theorem. So, $K$ has density $k$. \hfill \diamondsuit

Remark. If $p = 1$, then the relation may be written in terms of distribution functions as

$$K(z) = \int_{\mathbb{R}} G(z-x)F\{dx\}.$$ 

Example 3. If $F$ and $G$ are uniform on $(-\frac{1}{2}, \frac{1}{2})$, then $k(z) = (1 - |z|)^+$. \hfill \diamondsuit

Remark. $K$ is called the convolution of $F$ and $G$ and denoted by $K = F \ast G$. It follows directly from the proposition that $F \ast G = G \ast F$ and $(F \ast G) \ast H = F \ast (G \ast H)$ for distribution functions $F$, $G$, and $H$. Thus "$F \ast G \ast H$" is unambiguous. \hfill \diamondsuit

Proposition 6. More on Convolutions. If $F_1, \cdots, F_n$ are probability measures on $\mathbb{R}^p$, then $(F_1 \ast \cdots \ast F_n)(t) = \hat{F}_1(t) \times \cdots \times \hat{F}_n(t)$ for all $t$.

Proof. Construct independent random $p$-vectors $X_1, \cdots, X_n$ with distribution functions $F_1, \cdots, F_n$. Then $X_1 + \cdots + X_n$ has distribution function $F_1 \ast \cdots \ast F_n$ and, therefore,

$$(F_1 \ast \cdots \ast F_n)(t) = E[e^{it(X_1 + \cdots + X_n)}]$$

$$= E[e^{itX_1}] \times \cdots \times E[e^{itX_n}]$$

$$= \hat{F}_1(t) \times \cdots \times \hat{F}_n(t)$$

for all $t$. \hfill \diamondsuit

Example 1: Revisited. If $X_1, \cdots, X_n \sim^{\text{ind}} F$, where $F\{\pm 1\} = 1/2$, then the characteristic function of $S_n = X_1 + \cdots + X_n$ is $\cos^n(t)$.

Example 3: Revisited. If $X \sim \text{Unif}(-\frac{1}{2}, \frac{1}{2})$, then

$$E[e^{itX}] = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{itx}dx = \frac{\sin\left(\frac{1}{2}t\right)}{\frac{1}{2}t}.$$ 

So, if $X_1, \cdots, X_n \sim^{\text{ind}} \text{Unif}(-\frac{1}{2}, \frac{1}{2})$ and $S_n = X_1 + \cdots + X_n$, then

$$E[e^{its_n}] = \left(\frac{\sin\left(\frac{1}{2}t\right)}{\frac{1}{2}t}\right)^n.$$ 

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**Moments and Derivatives.** Recall that the moments of a distribution $F$ over $\mathbb{R}$ are $\mu_k = \int_{\mathbb{R}} x^k F(dx)$, if these exist. The same notation is used when $F$ is a finite measure. There is a simple relation between the moments of a distribution and derivatives of the characteristic function. Let

$$\rho_k(t) = e^{it} - \sum_{j=0}^{k-1} \frac{1}{j!} (it)^j$$

**Lemma 1.**

$$|\rho_k(t)| \leq \frac{1}{k!} |t|^k.$$  

**Proof.** If $k = 1$, then

$$\rho_1(t) = |e^{it} - 1| = |\int_0^t e^{is} ds| \leq \int_0^t ds = |t|,$$

Suppose that the Lemma were known for $k < m$, where $m \geq 2$. Then $\rho'_m(t) = i \rho_{m-1}(t)$, so that

$$|\rho_m(t)| = |\int_0^t \rho_{m-1}(s) ds| \leq \int_0^t |s|^{m-1} (m-1)! ds = \frac{|t|^m}{m!},$$

as asserted. \hfill \Box

**Lemma 2.** If $g$ is continuously differentiable in some neighborhood of 0 and $g''(0)$ exists, then

$$g''(0) = \lim_{s \to 0} \frac{g(s) + g(-s) - 2g(0)}{s^2}.$$  

**Proof.** Exercise.

**The Moments Theorem.** ($p = 1$). If $k \geq 1$ and $F$ has a finite $k^{th}$ moment, then $\hat{F}$ has $k$ continuous derivative given by

$$\hat{F}^j(t) = i^j \int_{\mathbb{R}} x^j e^{itx} F(dx)$$

(1)

for $t \in \mathbb{R}$ and $j = 1, \ldots, k$. Conversely, if $\hat{F}''(0)$ exist (finite), then $F$ has a finite second moment.

**Proof.** Suppose first that $j = 1$. If $h_n \to 0$, $h_n \neq 0$, and $t \in \mathbb{R}$, then

$$\frac{\hat{F}(t + h_n) - \hat{F}(t)}{h_n} = \int_{\mathbb{R}} g_n(x) F(dx),$$

where

$$g_n(x) = \frac{e^{ih_n x} - 1}{h_n} e^{itx}$$

for $x \in \mathbb{R}$. Clearly, $\lim_{n \to \infty} g_n(x) = xe^{itx}$ for all $x$, and $|g_n(x)| \leq |x|$ for all $n$ and $x$. So,

$$\lim_{n \to \infty} \int_{\mathbb{R}} g_n(x) F(dx) = \int_{\mathbb{R}} xe^{itx} F(dx),$$

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by the Dominated Convergence Theorem. It follows that \( \hat{F} \) is differentiable and that \( \hat{F}' \) is given by (1). That \( \hat{F}' \) is uniformly continuous, then follows from Proposition 1, since

\[
\hat{F}'(t) = i \int_{[0, \infty)} e^{itx} x F\{dx\} - i \int_{[0, \infty)} e^{itx} |x| F\{dx\},
\]

and both integrals are the Fourier transforms of finite measures. The general case \( j \geq 2 \) now follows by induction.

For the converse, if \( F \) is continuously differentiable on some neighborhood of 0 and \( F''(0) \) exists (finite), then

\[
-\hat{F}''(0) = \lim_{t \downarrow 0} \frac{2\hat{F}(0) - \hat{F}(t) - \hat{F}(-t)}{t^2}.
\]

The right side is

\[
\lim_{t \downarrow 0} 2 \int_{\mathbb{R}} \frac{1 - \cos(tx)}{t^2} F\{dx\} \geq \int_{\mathbb{R}} x^2 F\{dx\},
\]

by Fatou’s Lemma. \( \diamond \)

**Corollary.** If \( \mu_k \) is finite, then \( \hat{F}^{(k)}(0) = t^k \mu_k \).

**The Moments Theorem** (\( p > 1 \)). Let \( F \) be a finite measure over \( \mathbb{R}^p \) and let \( r_1, \ldots, r_p \) be non-negative integers. If

\[
\int_{\mathbb{R}^p} \left| \prod_{j=1}^p x_j^{r_j} F\{dx\} \right| < \infty,
\]

then

\[
\frac{\partial^{r_1 + \cdots + r_p}}{\partial t_1^{r_1} \cdots \partial t_p^{r_p}} \hat{F}(t_1, \ldots, t_p) = i^{r_1 + \cdots + r_p} \int_{\mathbb{R}^p} \prod_{j=1}^p x_j^{r_j} e^{it^j x} F\{dx\}
\]

is continuous.

**Proof.** Omitted.

**Normal Distributions.** Recall that the standard normal distribution function is

\[
\Phi(z) = \int_{(-\infty, z]} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} dy
\]

The normal distribution function with parameter \( \mu \in \mathbb{R} \) and \( \sigma^2 > 0 \) is

\[
F(x) = \Phi\left( \frac{x - \mu}{\sigma} \right).
\]

**Example: Univariate Normal Distributions.** If \( F \) is normal with parameters \( \mu \) and \( \sigma^2 \), then

\[
\hat{F}(t) = e^{i\mu t - \frac{1}{2} \sigma^2 t^2}
\]
for $t \in \mathbb{R}$.

**Proof.** Suppose first that $\mu = 0$ and $\sigma^2 = 1$, so that $F = \Phi$. Then

$$
\tilde{\Phi}''(t) = -\int_{-\infty}^{\infty} x^2 e^{itx} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \, dx = -\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^2 \cos(tx) e^{-\frac{1}{2}x^2} \, dx,
$$

by Moments Theorem. Integrating the right side by parts twice, then leads to

$$
\Phi''(t) = \frac{t^2 - 1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \cos(tx) e^{-\frac{1}{2}x^2} = (t^2 - 1)\Phi(t)
$$

for $t \in \mathbb{R}$. Moreover, $\Phi(0) = 1$ and $\Phi'(0) = 0$. The unique solution this differential equation is $\tilde{\Phi}(t) = e^{-\frac{1}{2}t^2}$, as asserted.

For the general case, let $Z \sim \Phi$ and $X = \sigma Z + \mu$. Then $X \sim F$ and, therefore, $\tilde{F}(t) = e^{i\mu t} \tilde{\Phi}(\sigma t)$.

**Corollary.** The mean and variance of $F$ are $\mu$ and $\sigma^2$

**The Mean Vector and Covariance Matrix.** If $X_1, \ldots, X_p$ have finite means, let $\mu_i = E(X_i)$. Then $\mu = (\mu_1, \ldots, \mu_p)'$ is called the mean vector of $X = (X_1, \ldots, X_p)'$. Similarly, if $X_i$ have finite variances, let $\sigma_{ij} = E[(X_i - \mu_i)(X_j - \mu_j)]$. Then $\Sigma = [\sigma_{ij}]$ is called the covariance matrix of $X = (X_1, \ldots, X_p)'$. These may be written as

$$
\mu = E(X),
\Sigma = E[(X - \mu)(X - \mu)'].
$$

**Proposition 7.** If $X$ has mean vector $\mu$ and covariance matrix $\Sigma$, and if $Y = AX + b$, where $A$ is $q \times p$ and $b \in \mathbb{R}^q$, then the mean vector and covariance matrix of $Y$ are $A\mu + b$ and $A\Sigma A'$, respectively.

**Proof.** Exercise.

**Multivariate Normal Distributions.** If $Z_1, \ldots, Z_p \sim \text{ind} \Phi$, then

$$
E[e^{it'Z}] = \ldots = e^{-\frac{1}{2}\|t\|^2}
$$

for $t \in \mathbb{R}^p$. So, if

$$
X = AZ + b,
$$

Then

$$
E[e^{it'X}] = e^{ib't} e^{-\frac{1}{2}\|A't\|^2} = e^{ib't - \frac{1}{2}t'AA't}.
$$

Alternatively,

$$
E[e^{it'X}] = e^{i\mu't - \frac{1}{2}t'\Sigma t},
$$

where $\mu$ and $\Sigma$ are the mean vector and covariance matrix of $X$. 

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