Guassian Processes
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The Mean and Covariance Functions. If $X_t, \quad t \in T,$ is a stochastic process for which $E(X_t^2) < \infty$ for all $t,$ then the mean and covariance functions are defined by

\[ m(t) = E(X_t) \]

and

\[ r(s,t) = C(X_s, X_t) \]

for all $s, t \in T.$ Thus, $r(t,t) = D^2(X_t),$ the variance of $X_t.$

Example: Sums. Let $Z_1, Z_2, \cdots$ be i.i.d. random variables with mean $E(Z_k) = 0$ and finite variance $E(Z_k^2) = \sigma^2.$ Also, let $g_1, g_2, \cdots$ be functions for which

\[ \sum_{k=1}^{\infty} g_k(t)^2 < \infty \]

for all $t \in T.$ Then

\[ X_t = \sum_{k=1}^{\infty} Z_k g_k(t) \]

converges with probability one and in $L^2$ for each $t.$ Let $X_t = 0,$ if the sum does not converge. From the convergence in $L^2,$ it is easily seen that

\[ r(s,t) = \sigma^2 \sum_{k=1}^{\infty} g_k(s) g_k(t). \]

In fact, letting $X_{n,t} = Z_1 g_1(t) + \cdots + Z_n g_n(t)$ and $r_n(s,t) = g_1(s) g_1(t) + \cdots + g_n(s) g_n(t),$ we have

\[ E[X_{n,s} X_{n,t} - X_s X_t] = E[(X_{n,s} - X_s) X_{n,t} (X_{n,t} - X_t) X_s] + E(X_{n,s} - X_s) (X_{n,t} - X_t), \]

which approaches 0 as $n \to \infty,$ by Schwarz’ Inequality.

**Proposition 1.** If $r$ is the covariance function of a process $X_t, \quad t \in T,$ then

\[ r(t,s) = r(s,t) \quad (1) \]

and

\[ \sum_{i=1}^{m} \sum_{j=1}^{m} a_i a_j r(t_i,t_j) \geq 0 \quad (2) \]

for all $s, t, t_1, \cdots, t_m \in T$ and $a_1, \cdots, a_m \in R.$
Proof. The first assertion is clear. Given \( a_1, \ldots, a_m \in \mathbb{R} \) and \( t_1, \ldots, t_m \in T \), let \( S = a_1X_{t_1} + \cdots + a_mX_{t_m} \). Then

\[
0 \leq D^2(S) = \sum_{i=1}^m \sum_{j=1}^m a_ia_j r(t_i, t_j),
\]

as asserted. \( \diamond \)

Non-negative Definite Functions. Any function \( r : T \times T \rightarrow \mathbb{R} \) for which (1) and (2) hold is said to be non-negative definite. It is clear that \( r \) is non-negative definite iff its restriction to \( J \times J \) is non-negative definite for every finite subset \( J \subseteq T \). Further, if \( T \) is finite, say \( T = \{1, \ldots, m\} \), then \( r \) is non-negative definite iff

\[
\Sigma = \begin{pmatrix} r(1,1) & \cdots & r(1,m) \\ \cdots & \cdots & \cdots \\ r(m,1) & \cdots & r(m,m) \end{pmatrix}
\]

is a symmetric, non-negative definite matrix.

Example 1. If \( T = [0, \infty) \), then \( r(s,t) = s \wedge t \) is a non-negative definite function. To see this, let \( Z_1, Z_2, \ldots \sim \text{i.i.d.} \) standard normal random variables; let \( 0 \leq t_1 < \cdots < t_m < \infty \); and let

\[
Y_j = \sum_{i=1}^j \sqrt{(t_i - t_{i-1})} Z_i,
\]

for \( j = 1, \ldots, m \). Then

\[
C(Y_j, Y_k) = \sum_{i=1}^{j \wedge k} (t_i - t_{i-1}) = t_j \wedge t_k
\]

for \( j, k = 1, \ldots, m \), so that the restriction of \( r \) to \( \{t_1, \cdots, t_m\} \times \{t_1, \cdots, t_m\} \) is non-negative definite. \( \diamond \)

Notation. Until further notice, \( \mathbb{R}^m \) is the set of \( m \times 1 \) matrices \( x = [x_1, \cdots, x_m]' \), where \( ' \) denotes transpose. Then

\[
x'x = x_1^2 + \cdots + x_m^2 = \|x\|^2,
\]

the Euclidean norm of \( x \).

Multivariate Normal Distributions. If \( Z_1, \cdots, Z_m \sim \text{i.i.d.} \) are independent standard normal random variables, then the characteristic function of the random vector \( Z = (Z_1, \cdots, Z_m)' \) is

\[
E[e^{is'Z}] = E\left[ \prod_{j=1}^m e^{is_jZ_j} \right] = \prod_{j} e^{-\frac{1}{2}s_j^2} = e^{-\frac{1}{2}\|s\|^2}
\]
for $s \in \mathbb{R}^m$. Next if 

$$X = AZ + b,$$

where $A$ is a $m \times m$ matrix and $b \in \mathbb{R}^m$, then the characteristic function of $X$ is 

$$E[e^{is'X}] = e^{is'b} E[e^{is'AZ}] = e^{is'b - \frac{1}{2}s'AA's}.$$ 

Differentiation then shows that $E(X_j) = b_j$ and $C(X_i, X_j) = (AA')_{ij}$ for all $i$ and $j$. Accordingly, 

$$\mu = b \quad \text{and} \quad \Sigma = AA'$$

are called the mean vector and covariance matrix of $X$, and the characteristic function is written as 

$$E[e^{is'X}] = e^{is'b} E[e^{is'AZ}] = e^{is'\mu - \frac{1}{2}s'\Sigma s}.$$ 

The distribution of $X$ is called the multivariate normal distribution with mean vector $\mu$ and covariance matrix $\Sigma$. Since characteristic functions uniquely determine distribution functions, the distribution of $X$ depends only on $\mu$ and $\Sigma$.

**Proposition 2.** If $Y \sim \text{Normal}_m(\mu, \Sigma)$ and $Y = AX + b$, where $A$ is $n \times m$ and $b \in \mathbb{R}^n$, then $Y \sim \text{Normal}_n[A\mu + b, A\Sigma A']$.

**Proof.** Exercise—similar to the derivation.

**Corollary 1.** If $X = (X_1, \ldots, X_m)$ has a multivariate normal distribution and if $1 \leq k < m$, then $(X_1, \ldots, X_k)$ has a multivariate normal distribution.

**Proof.** Take $A = [I_k, 0]$ and $b = 0$. ◊

**Corollary 2.** Suppose that 

$$X = \begin{pmatrix} X^1 \\ X^2 \end{pmatrix} \sim \text{Normal}_m[ \begin{pmatrix} \mu^1 \\ \mu^2 \end{pmatrix}, \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix} ],$$

where $X^1 = (X_1, \ldots, X_k)'$ and $X^2 = (X_{k+1}, \ldots, X_m)'$ and $1 \leq k < m$. Then $X^1$ and $X^2$ are independent iff $\Sigma_{12} = 0$.

**Proof.** The condition is obviously necessary. Let $Q$, $Q^1$, and $Q^2$ denote the distributions of $X$, $X^1$, and $X^2$. If $\Sigma_{12} = 0$, then 

$$\hat{Q}(s^1, s^2) = \cdots = \hat{Q}(s^1)\hat{Q}(s^2)$$

for all $s \in \mathbb{R}^m$. So, $X^1$ and $X^2$ are independent. ◊

**Remark.** Corollary 2 extends to more than two subvectors in a straightforward way.

**Corollary 3.** $X \sim \text{Normal}_m(\mu, \Sigma)$ iff $s'X \sim \text{Normal}(s'm, s'\Sigma s)$ for all $s \in \mathbb{R}^m$.

**Proof.** The only if assertion is clear; and if $s'X \sim \text{Normal}(s'm, s'\Sigma s)$, then 

$$\hat{Q}(s) = E[e^{is'X}] = \hat{Q}_s(1) = e^{is'\mu - \frac{1}{2}s'\Sigma s},$$

as required. ◊

**Gaussian Processes.** A stochastic process $X_t$, $t \in T$, is called a Gaussian process iff $(X_{t_1}, \ldots, X_{t_m})'$ has a multivariate normal distribution for every choice of $t_1, \ldots, t_m \in T$. There are two important existence Theorems.
**Theorem 1.** Given any set $T$, and real valued function $m : T \to \mathbb{R}$, and any non-negative definite function $r : T \times T \to \mathbb{R}$, there is a Gaussian process with mean function $m$ and covariance function $r$.

*Proof-Outline.* For each list $t = (t_1, \cdots, t_m)$ of distinct elements of $T$, let $Q$ be the multivariate normal distribution with mean vector and covariance matrix

$$
\begin{pmatrix}
m(t_1) \\
\vdots \\
m(t_n)
\end{pmatrix}
\quad \text{and} \quad
\begin{pmatrix}
r(t_1, t_1) & \cdots & r(t_1, t_m) \\
\vdots & \ddots & \vdots \\
r(t_m, t_1) & \cdots & r(t_m, t_m)
\end{pmatrix}.
$$

The Komogorov’s Consistency Conditions are satisfied by an easy application of Proposition 2 above. The existence of the process then follows from Kolmogorov’s Consistency Theorem.

**Theorem 2:** Karhunen-Loeve Expansions. Let $T$ be any set; let $g_k : T \to \mathbb{R}$, $k = 1, 2, \cdots$ be function for which $g_1(t)^2 + g_2(t)^2 + \cdots < \infty$ for all $t \in T$; and let $Z_1, Z_2, \cdots \sim \text{ind } \Phi$. Then

$$X_t = \sum_{k=1}^{\infty} Z_k g_k(t)$$

is a Gaussian process with mean $m = 0$ and covariance function

$$r(s,t) = \sum_{k=1}^{\infty} g_k(s)g_k(t).$$

*Proof.* The series defining $X_t$ converges $w.p.1$ and in mean square for all $t$, by the three series theorem. Let $X_t = 0$, if the series does not converge. Then $X_t, t \in T$, is a stochastic process with mean $m = 0$ and covariance function $r$. So, it remains only to show that $X_t, t \in T$, is a Gaussian process. Let

$$X_{n,t} = \sum_{k=1}^{n} Z_k g_k(t)$$

for $t \in T$. Then each $X_{n,t}, t \in T$, is a Gaussian process, by Proposition 2. For if $t_1, \cdots, t_m \in T$, then $[X_{n,t_1}, \cdots, X_{n,t_m}]'$ is obtained by a linear transformation of $Z_1, \cdots, Z_n$. Given $t_1, \cdots, t_m \in T$, let $\Sigma_n$ be the covariance matrix of of $Y_n[X_{n,t_1}, \cdots, X_{n,t_m}]'$ and let $\Sigma$ be the covariance matrix of $Y = [X_{t_1}, \cdots, X_{t_m}]'$. Then

$$E[e^{i\theta'}Y] = \lim_{n \to \infty} E[e^{i\theta'Y_n}] = \lim_{n \to \infty} e^{-\frac{1}{2}\theta'\Sigma_n\theta} = e^{-\frac{1}{2}\theta'\Sigma\theta},$$

so that $Y$ has a multivariate normal distribution.

*Remark.* If $T$ is an interval, $g_1, g_2, \cdots$ are continuous, and

$$\lim_{n \to \infty} \sup_{t \in T} |X_{n,t} - X_t| = 0 \text{ w.p.1},$$

then $X_t, t \in T$, has continuous sample paths. Relation (3) does not follow from the Two Series Theorem, however.

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Lemma 1. For $0 \leq s, t \leq 1$,

$$s \wedge t = st + \frac{2}{\pi} \sum_{k=1}^{\infty} \frac{\sin(k\pi s) \sin(k\pi t)}{k^2}.$$  

Proof-Outline. For a fixed $s \in [0, 1]$, let $h_s(t) = s \wedge t - st$ for $0 \leq t \leq 1$, and $h_s(-t) = -h_s(t)$. Then

$$h_s(t) = \sum_{k=1}^{\infty} c_k \sin(k\pi t),$$

where

$$c_k = \int_{-1}^{1} \sin(k\pi u) h_s(u) du = \cdots = \frac{2 \sin(k\pi s)}{\pi k^2}.$$  

Examples 2-a): Brownian Motion. Let $Z_0, Z_1, Z_2, \cdots \sim \text{ind } \Phi$ and

$$\mathbb{B}_t = Z_0 t + \sum_{k=1}^{\infty} \frac{1}{k} Z_k \sin(k\pi t)$$

for $0 \leq t \leq 1$. Then $\mathbb{B}_t$, $0 \leq t \leq 1$, is a Gaussian process with mean 0 and covariance function $r(s, t) = s \wedge t$. It follows that for $0 \leq s < t \leq 1$, $E[(\mathbb{B}_t - \mathbb{B}_s)^2] = t-s$, $E[(\mathbb{B}_t - \mathbb{B}_s)^4] = 3(t-s)^2$ and, therefore, that $\mathbb{B}$ has a continuous modification. Any such continuous modification is called standard Brownian motion on $[0, 1]$.

b): The Brownian Bridge. Similarly,

$$\mathbb{B}_t^0 = \sum_{k=1}^{\infty} \frac{1}{k} Z_k \sin(k\pi t), 0 \leq t \leq 1,$$

is a Guassian process with mean 0 and covariance function $r_0(s, t) = s \wedge t$. As above, $E[(\mathbb{B}_t^0 - \mathbb{B}_s^0)^2] \leq t-s$ and, therefore, that $\mathbb{B}_t^0$ has a continuous modification. Any such continuous modification is called standard Brownian bridge.  

Continuity. The following lemma is needed for the existence of continuous modifications.

Lemma 2. Let $\varphi$ and $\Phi$ denote the standard normal density and distribution function. Then

$$1 - \Phi(x) \leq \frac{1}{x} \varphi(x)$$

for $0 < x < \infty$.

Proof. Since $\varphi'(y) = -y \varphi(y)$,

$$1 - \Phi(x) = \int_{[x, \infty)} \varphi(y) dy \leq \frac{1}{x} \int_{[x, \infty)} y \varphi(y) dy = \frac{1}{x} \varphi(x)$$

for all $x > 0$.  

\[ \square \]
Theorem 3. Let $T$ is an interval and let $X_t$, in $T$, be a Gaussian process with mean function $m$ and covariance function $r$. If $m$ is continuous and there are constants $0 < \alpha, C$ for which
\[
r(s, s) - 2r(s, t) + r(t, t) \leq C \frac{1}{\log \left[ \frac{1}{|t-s|} \right]^{3+3\alpha}}
\]
for $|t-s| \leq 1$, then $X$ has a continuous modification.

Proof-Outline. For the existence of a continuous modification, there is no loss of generality in supposing that $T$ is compact and that $m = 0$. Let
\[
\eta(\delta) = \left[ \frac{1}{\log \left( \frac{1}{\delta} \right)} \right]^{1+\alpha},
\]
for $0 < \delta \leq 1$. Then
\[
\sum_{k=1}^{\infty} \eta(2^{-k}) \geq \sum_{k=1}^{\infty} \left[ \frac{1}{k \log(2)} \right]^{1+\alpha} < \infty.
\]
If $s, t \in T$ and $0 \leq t - s \leq \delta$, then $X_t - X_s \sim \text{Normal}[0, \sigma^2]$, where $\sigma^2 = r(s, s) - 2r(s, t) + r(t, t)$. So,
\[
P[|X_t - S_s| \geq \eta(\delta)] = 2 \{1 - \Phi \left[ \frac{\eta(\delta)}{\sigma} \right] \}.
\]
Here
\[
\frac{\eta(\delta)^2}{\sigma^2} \geq C \left[ \log \left( \frac{1}{\delta} \right) \right]^{1+\alpha} \geq 4 \log \left( \frac{1}{\delta} \right) \geq 1
\]
for all sufficiently small $\delta$. So,
\[
P[|X_t - S_s| \geq \eta(\delta)] \leq 2 \exp \left[ -\frac{1}{2} \frac{\eta(\delta)^2}{\sigma^2} \right] \leq 2 \exp [2 \log(\delta)] \leq \delta^2
\]
for all $|t - s| \leq \delta$ fora ll small $\delta$. The existence of a continuous modification now follows from an earlier theorem with $\xi(\delta) = \delta^2$.

Corollary. If there are constants $0 < \alpha, C$ for which $r(s, s) - 2r(s, t) + r(t, t) \leq C|t - s|^\alpha$, then $X$ has a continuous modification.

Problem. Let $X_t, a \leq t \leq b$, be a Gaussian process with continuous sample paths, a continuous mean function $m$, and a continuous covariance function $r$. Show that
\[
J(\omega) = \int_a^b X_t(\omega) d\omega
\]
define a (measurable) random variable. Show further that $J$ has a normal distribution and express the mean and variance of $J$ in terms of $m$ and $r$. 

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