The Martingale Convergence Theorem II
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Uniform Integrability. Recall: A sequence of random variables $X_n$, $n \geq 1$, defined on a probability space $(\Omega, \mathcal{A}, P)$, say, is uniformly integrable if

$$
\lim_{c \to \infty} \sup_{n \geq 1} \int_{\{|X_n| > c\}} |X_n| dP = 0. \quad (1)
$$

Thus, $X_n$ are uniformly integrable iff $|X_n|$ are. Two main results about uniform integrability are needed. First $X_n$ are uniformly integrable iff

$$
\sup_n E|X_n| < \infty \quad (2)
$$

and

$$
\lim_{P(A) \to 0} \sup_n \int_A |X_n| dP = 0. \quad (3)
$$

Here the meaning of (3) is that for every $\epsilon > 0$ there is a $\delta > 0$ for which $\int_A |X_n| dP \leq \epsilon$ for all $n$ whenever $A \in \mathcal{A}$ and $P(A) \leq \delta$. Second, if $X_n \to^p X_0$ and $E|X_n| < \infty$ for all $n \geq 0$, then

$$
\lim_{n \to \infty} E|X_n - X_0| = 0
$$

iff $X_n$, $n \geq 0$ are uniformly integrable.

Example. If $\mathcal{A}_n$, $n \geq 1$, are sigma-algebra and $Y$ is an integrable random variable, then

$$
Y_n = E(Y | \mathcal{A}_n), \ n \geq 1,
$$

are uniformly integrable.

Proof. Clearly, $E|Y_n| \leq E|Y|$ for all $n \geq 1$. So, $P\{|Y_n| > c\} \leq E|Y|/c$ for all $n \geq 1$ and $c > 0$. Next,

$$
\int_{\{|Y_n| > c\}} |Y_n| dP \leq \int_{\{|Y_n| > c\}} E(|Y| | \mathcal{A}_n) dP = \int_{\{|Y_n| > c\}} |Y| dP,
$$

since $\{|Y_n| > c\} \in \mathcal{A}_n$. Given $\epsilon > 0$, there is a $\delta$ for which $\int_A |Y| dP \leq \epsilon$ whenever $P(A) \leq \delta$. So, if $c > E|Y|/\delta$, then $P\{|Y_n| > c\} \leq \delta$ for all $n$ and, therefore,

$$
\int_{\{|Y_n| > c\}} |Y_n| dP \leq \int_{\{|Y_n| > c\}} |Y| dP \leq \epsilon
$$

for all $n$, as required. \qed

Problem 1. If $X_n \geq 0$ have distribution functions $F_n$ then $X_n$ are uniformly integrable iff

$$
\lim_{c \to \infty} \sup_n \int_{-\infty}^{\infty} [1 - F_n(x)] dx = 0.
$$

In particular, $X_n$ are uniformly integrable if $\int_0^\infty \sup_n [1 - F_n(x)] dx < \infty$. 

1
Recall the Submartingale Convergence Theorem: If $Y_0, Y_1, Y_2, \cdots$ is a submartingale for which $K = \sup_n E|Y_n| < \infty$, then $Y = \lim_{n \to \infty} Y_n$ exists w.p.1 and $E|Y| \leq K$.

**Corollary 1.** If $Y_0, Y_1, \cdots$ is a uniformly integrable submartingale, then $Y_\infty = \lim_{n \to \infty} Y_n$ exists w.p.1 and $\lim_{n \to \infty} E|Y_n - Y_\infty| = 0$.

**Proof.** This is clear.

**Corollary 2.** If $Y_0, Y_1, Y_2, \cdots$ is uniformly integrable submartingale with respect to $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \cdots$, then $Y_m \leq E(Y_\infty | \mathcal{A}_m)$ w.p.1 for all $m \geq 0$, and there is equality in the martingale case.

**Proof.** It suffices to show that
\[ \int_A Y_m dP \leq \int_A Y_\infty dP \] (4)
for all $A \in \mathcal{A}_m$ with equality in the martingale case. From the submartingale property,
\[ \int_A Y_m dP \leq \int_A Y_n dP \] (5)
for all $A \in \mathcal{A}_m$ and $n \geq m$ with equality in the martingale case. Relation (4) then follows by letting $n \to \infty$ in (5) and using the uniform integrability. \hfill \Box

**The Join of Sigma-Algebra.** Recall that the join of sigma-algebras $\mathcal{A}_1, \mathcal{A}_2, \cdots$ is the smallest sigma-algebra containing all of $\mathcal{A}_1, \mathcal{A}_2, \cdots$; in symbols,
\[ \bigvee_{n=1}^\infty \mathcal{A}_n = \sigma \{ \bigcup_{n=1}^\infty \mathcal{A}_n \}. \]
If $\mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$, then the join is denoted by $\mathcal{A}_\infty$.

**Theorem 1.** Let $Y$ be an integrable random variable. If $\mathcal{A}_0 \subset \mathcal{A}_1 \subset \mathcal{A}_2 \subset \cdots$, then
\[ \lim_{n \to \infty} E(Y | \mathcal{A}_n) = E(Y | \mathcal{A}_\infty) \text{ w.p.1.} \]

**Proof.** There is no loss of generality in supposing that $Y \geq 0$. Let $Y_n = E(Y | \mathcal{A}_n)$ for $n \geq 1$. Then $Y_n$ is a uniformly integrable martingale. So, $Y_\infty = \lim_{n \to \infty} Y_n$ exists w.p.1, and it suffices to show that $Y_\infty = E(Y | \mathcal{A}_\infty)$. Since $Y_\infty$ is $\mathcal{A}_\infty$ measurable, this means that
\[ \int_A Y_\infty dP = \int_A Y dP \] (6)
for all $A \in \mathcal{A}_\infty$. If $A \in \mathcal{A}_m$ for some $m$, then $\int_A Y_n dP = \int_A Y dP$ for all $n \geq m$ and, therefore,
\[ \int_A Y_\infty dP = \lim_{n \to \infty} \int_A Y_n dP = \int_A Y dP. \]
Thus (6) holds for all $A \in \bigcup_{n=0}^\infty \mathcal{A}_n$, an algebra, and since both sides of (6) define measures, (6) must hold of all $A \in \sigma \{ \bigcup_{n=0}^\infty \mathcal{A}_n \} = \mathcal{A}_\infty$. \hfill \Box
A Constructive Approach to Conditional Expectation

**Corollary 1.** Let $\mathcal{B} \subseteq \mathcal{A}$ and suppose that there are partitions $\mathcal{C}_n = \{C_{n1}, \ldots, C_{nk_n}\}$ for which $\mathcal{B}_n = \sigma\{C_n\} \uparrow \mathcal{B}$. If $Y$ is integrable, then

$$E(Y|\mathcal{B}) = \lim_{n \to \infty} \sum_{k=1}^{k_n} \left[ \frac{1}{P(C_{nk})} \int_{C_{nk}} Y \, dP \right] 1_{C_{nk}} \text{ w.p.1.}$$

**Proof.** This is clear, since $E(Y|\mathcal{B}_n) \to E(Y|\mathcal{B})$ w.p.1, and

$$E(Y|\mathcal{B}_n) = \sum_{k=1}^{k_n} \left[ \frac{1}{P(C_{nk})} \int_{C_{nk}} Y \, dP \right] 1_{C_{nk}} \text{ w.p.1}$$

for each $n$. \hfill \(\square\)

**Corollary 2.** Let $X$ and $Y$ be random variables for which $E|Y| < \infty$, and let $J_{nk} = ((k - 1)/2^n, k/2^n]$ and

$$e_n = \sum_{k \in \mathbb{Z}} \left[ \int_{X \in J_{nk}} Y \, dP \frac{X}{P(X \in J_{nk})} \right] 1_{J_{nk}}.$$  

Then

$$\lim_{n \to \infty} e_n(x) = E(X|Y)$$

for a.e. $x$ (F), the distribution of $X$.

**Proof.** Exercise

**An Application to Radon Nykodym Derivatives.** Now let $\nu$ be a finite measure on $\mathcal{A}$ for which $\nu \ll P$ and suppose that there are partitions $\mathcal{C}_n = \{C_{n1}, \ldots, C_{nk_n}\}$, $n \geq 1$, for which $\mathcal{B}_n = \sigma\{C_n\} \uparrow \mathcal{A}$. Let

$$R_n = \sum_{k=1}^{k_n} \left[ \frac{\nu(C_{nk})}{P(C_{nk})} \right] 1_{C_{nk}},$$

where $0/0$ is to be interpreted as 0.

**Corollary 3.** With the notation of the previous paragraph,

$$\lim_{n \to \infty} R_n = \frac{d\nu}{dP} \text{ w.p.1.}$$

**Proof.** Denote the Radon Nykodym derivative by $R = d\nu/dP$. Then

$$R_n = E(R|\mathcal{A}_n) \text{ w.p.1}$$

for each $n$, since $R_n$ is $\mathcal{A}_n$-measurable for each $n$ and

$$\int_{C_{nj}} R_n dP = \nu(C_{nj}) = \int_{C_{nj}} RdP$$

for each $n$.
for all $j$ and $n$. So,

$$
\lim_{n \to \infty} R_n = E(R|A_\infty) = E(R|A) = R \; w.p.1
$$

as asserted. \hfill \diamond

**Example.** Suppose that $\Omega = [0,1]$ and $P = \lambda$ is the restriction of Lebesgue measure to the Borel sets of $[0,1]$. If $\nu$ is a finite measure and $\nu \ll \lambda$, then

$$
\frac{d\nu}{d\lambda} = \lim_{n \to \infty} \sum_{k=1}^{2^n} 2^n \nu\left(\left(\frac{k-1}{2^n}, \frac{k}{2^n}\right]\right) \mathbf{1}_{J_{nk}} \; w.p.1,
$$

where $J_{nk} = ((k-1)/2^n, k/2^n]$, as above. \hfill \diamond

**Problem 2.** Let $X_0, X_1, X_2, \cdots$ be i.i.d. random variables that are uniformly distributed over $[0,1]$. Further, let

$$
W_k = \mathbf{1}\{X_k \leq x_0\}, \quad A_n = \sigma\{W_1, \cdots, W_n\}.
$$

Show that

$$
\lim_{n \to \infty} E(X_0|A_n) = X_0 \; w.p.1.
$$

**Reverse Martingales.** If $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ is a decreasing sequence of sigma-algebra of $\mathcal{A}$, then a sequence $Y_0, Y_1, Y_2, \cdots$ is said to be a reverse (sub) (super) martingale with respect to $A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots$ iff the following hold for all $n \geq 0$:

a) $Y_n$ is $A_n$-measurable;

b) $E|Y_n| < \infty$;

c) $E(Y_n|A_{n+1}) (\geq) (\leq) = Y_n \; w.p.1$.

**Remark i.** Then, for any $m$,

$$
Z_{m,n} = \begin{cases} 
Y_{m-n} & \text{if } 0 \leq n \leq m \\
Y_0 & \text{if } n > m 
\end{cases}
$$

is a (sub) (super) martingale with respect to $A_{m,n} = A_{(m-n)^+}$.

**ii.** So, if $U_n(\alpha, \beta)$ is the number of upcrossings of the interval $[\alpha, \beta]$ by $Y_0, \cdots, Y_n$, then

$$
E[U_n(\alpha, \beta)] \leq 1 + \frac{E[(Y_0 - \alpha)^+]}{\beta - \alpha}.
$$

**Example.** If $A_0 \supseteq A_1 \supseteq \cdots$ and $E|Y| < \infty$, then $Y_n = E(Y|A_n)$ is a reverse martingale, by the smoothing property of conditional expectation. \hfill \diamond

**Lemma.** If $A_0 \supseteq A_1 \supseteq \cdots$ and $Y_n$ is $A_n$-measurable for each $n$, then $\liminf_{n \to \infty} Y_n$ is measurable with respect to $A_\infty = \cap_{n=0}^\infty A_n$.

**Proof.** Exercise.
**Theorem 2** The Reverse Martingale Convergence Theorem. If $Y_0, Y_1, Y_2, \cdots$ is a reverse martingale, then $Y_n = \lim_{n \to \infty} Y_n$ exists w.p.1, is integrable, and $\lim_{n \to \infty} E|Y_n - Y| = 0$.

**Proof:** Outline. The existence of the limit follows from the upcrossings lemma, as in the proof of the Submartingale Convergence Theorem. Letting $\mathcal{B}_n = \sigma\{Y_0, \cdots, Y_n\}$, uniform integrability then follows from $|Y_n| = |E(Y_0|\mathcal{B}_n)| \leq E(|Y_0|\mathcal{B}_n)$ w.p.1 for all $n$. \hfill \Diamond

**Corollary.** Let $\mathcal{A}_0 \supseteq \mathcal{A}_1 \supseteq \cdots$ and $\mathcal{A}_\infty = \cap_{n=0}^{\infty} \mathcal{A}_n$. If $E|Y| < \infty$, then

$$
\lim_{n \to \infty} E(Y|\mathcal{A}_n) = E(Y|\mathcal{A}_\infty) \text{ w.p.1.}
$$

**Proof.** $Y_n$ is a uniformly integrable reverse martingale. So, $Y_\infty = \lim_{n \to \infty} Y_n$ exists w.p.1, $Y_\infty$ is $\mathcal{A}_\infty$-measurable, and $\lim_{n \to \infty} E|Y_n - Y_\infty| = 0$. If $A \in \mathcal{A}_\infty$, then $A \in \mathcal{A}_m$ for every $m$ and, therefore,

$$
\int_A Y_\infty dP = \lim_{n \to \infty} \int_A Y_n dP = \int_A Y dP,
$$

as required. \hfill \Diamond

**The Strong Law of Large Numbers: An Outline.** Let $X_1, X_2, \cdots$ be i.i.d. with a finite mean $\mu$; and let $S_n = X_1 + \cdots + X_n$.

**Lemma.**

$$
E(X_1|S_n) = \frac{S_n}{n}.
$$

**Proof.** By symmetry $E(X_i|S_n) = E(x_1|S_n)$ for all $i$ and, therefore,

$$
E(X_1|S_n) = \frac{1}{n} \sum_{i=1}^{n} E(X_i|S_n) = \frac{1}{n} E(S_n|S_n) = \frac{S_n}{n}.
$$

**Lemma.** Let $\mathcal{B}$ and $\mathcal{C}$ be sigma-algebra; and let $Y$ be a random variable for which $E|Y| < \infty$ and $\mathcal{B} \vee \sigma(Y)$ is independent of $\mathcal{C}$. Then

$$
E(Y|\mathcal{B} \vee \mathcal{C}) = E(Y|\mathcal{B}).
$$

**Proof.** The lemma asserts that

$$
\int_D Y dP = \int_D E(Y|\mathcal{B}) dP \tag{(*)}
$$

for all $D \in \mathcal{B} \vee \mathcal{C}$. If $D = B \cap C$, where $B \in \mathcal{B}$ and $C \in \mathcal{C}$, then (*) is clear, since then

$$
\int_D Y dP = P(C) \int_B Y dP = P(C) \int_B E(Y|B) dP = \int_B E(Y|D) dP,
$$

as required. \hfill \Diamond
The Zero-One Law. If $X_1, X_2, \cdots$ are independent and $\mathcal{T}_n = \sigma\{X_{n+1}, X_{n+2}, \cdots\}$, then $\mathcal{T}_\infty = \cap_{n=0}^\infty \mathcal{T}_n$ is trivial.

**Proof.** See the text.

The Strong Law of Large Numbers.

$$\lim_{n \to \infty} \frac{S_n}{n} = \mu \text{ w.p.1.}$$

**Proof: Outline.** Let

$$\mathcal{A}_n = \sigma\{S_n, X_{n+1}, X_{n+2}, \cdots\} = \sigma\{S_n\} \vee \sigma\{X_{n+1}, X_{n+2}, \cdots\}.$$ 

Then

$$E(X_1|\mathcal{A}_n) = E(X_1|S_n) = \frac{S_n}{n}.$$ 

So,

$$Y = \lim_{n \to \infty} \frac{S_n}{n} = E(X_1|\mathcal{A}_\infty)$$

and

$$E(Y) = \mu.$$ 

But $Y$ is constant w.p.1, by the Zero One Law and, therefore, $Y = \mu \text{ w.p.1.} \quad \diamondsuit$