Estimating Dark Matter

Distributions

In Progress

By

Xaio Wang

Mathew Walker

Mario Mateo

Michael Woodroofe

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Objects of Study

The Milky Way

Some Nearby Dwarves

Draco

Fornax

Leo I and II

...

Ursa Minor

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Dark Matter

Existence: If all matter were visible, then stars wouldn't be where they are or move as they do!

What is it?

- Don't Know
- Exclusions–e.g. dead stars.

Where it is?

- This is our objective.
- May shed light on what it is.

Inverse Problem: The mass distribution determines positions and velocities.

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Some Potential Theory

If ρ is a mass density in $\mathbb{R}^3$, let

$$\Phi(x) = -G \int \frac{\rho(\boldsymbol{y})\boldsymbol{d}y}{\|\boldsymbol{x}-\boldsymbol{y}\|},$$

where G is the gravitational constant,

$$\boldsymbol{x} = (x_1, x_2, x_3)$$

and

$$\|\boldsymbol{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Then the force field is

$$\mathbf{F}(\boldsymbol{x}) = G \int \frac{(\boldsymbol{y}-\boldsymbol{x})\rho(\boldsymbol{y})\boldsymbol{d}y}{\|\boldsymbol{x}-\boldsymbol{y}\|^3} = -\nabla\Phi(x),$$

and

$$4\pi G\rho(\boldsymbol{x}) = \Delta\Phi(\boldsymbol{x}),$$

where ∇ and Δ are gradient and Laplacian–.e.g.

$$\Delta\Phi(\boldsymbol{x}) = \frac{\partial^2\Phi(\boldsymbol{x})}{\partial x_1^2} + \frac{\partial^2\Phi(\boldsymbol{x})}{\partial x_2^2} + \frac{\partial^2\Phi(\boldsymbol{x})}{\partial x_3^2}.$$

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The Data

Projected Positions

- From Photographs
- Easier: Large Samples.

Radial Velocities

- From redshifts
- Harder: Need time to get enough light.
- Small samples–.e.g. 30.
- Parametric models.

What's ν:

- Better mousetrap: larger samples.
- Can do non-parametric analysis.

Notation

Positions and Vecocities of Stars

For a dwarf-spheroidal galaxy, let

$$\boldsymbol{X} = (X_1, X_2, X_3)$$

and

$$\boldsymbol{V} = (V_1, V_2, V_3)$$

denote the position and velocity of a star. Regard these as random vectors and suppose that they have a distribution of the form

$$P\big[\boldsymbol{x} \le \boldsymbol{X} \le \boldsymbol{x} + \boldsymbol{dx},\ \boldsymbol{v} \le \boldsymbol{V} \le \boldsymbol{v} + \boldsymbol{dv}\big] = f_0(r, v)\boldsymbol{dx}\boldsymbol{dv},$$

where

$$r^2 = x_1^2 + x_2^2 + x_3^2$$
$$v^2 = v_1^2 + v_2^2 + v_3^2$$

Here $\mathbf{0} = (0, 0, 0)$ is the center of the galaxy, and

$$\int \boldsymbol{v} f_0(r, v)\boldsymbol{dx}\boldsymbol{dv} = \mathbf{0}.$$

The Mass Distribution

Let ρ denote the mass density, Suppose that ρ is spherically symmetric, and let

$$M(r) = 4\pi \int_0^r t^2 \rho(t) dt,$$

the total mass in the sphere of radius r, centered at the origin.

Goal: Estimate M from data.

Jean's Equation: From potential theory,

$$M(r) = -\frac{r^2 \mu(r)}{G} \frac{d}{dr} \log\big[f(r)\mu(r)\big],$$

where

$$f(r) = \int f_0(r, v)\boldsymbol{dv},$$

and

$$\mu(r) = \frac{1}{3f(r)} \int v^2 f_0(r, v)\boldsymbol{dv}$$

Note: In Astronomy, $\mu(r)$ is called the *velocity dispersion.*

Statistically:

$$P\big[\boldsymbol{x} \le \boldsymbol{X} \le \boldsymbol{x} + \boldsymbol{d}x\big] = f(r)\boldsymbol{dx}$$

and

$$\mu(r) = E\Big[\frac{V_1^2 + V_2^2 + V_3^2}{3} | \boldsymbol{X} = \boldsymbol{x}\Big].$$

Incomplete Observation: Observe

- Projected Positions
- Radial Velocity (from the red shift).

With a proper choice of cordinates, these are X_1, X_2, and V_3. Here

$$\mu(r) = E\big[V_3^2 | \boldsymbol{X} = \boldsymbol{x}\big],$$

but missing X_3 causes more serious problems, known as Wicksell's (1925, *Biometrika*) Problem.

Estimating f

Let

$$R^2 = X_1^2 + X_2^2 + X_3^2$$

and

$$S^2 = X_1^2 + X_2^2$$

and let $f_{\mathbf{r}}$ and $g_{\mathbf{s}}$ denote the densities of these random variables, so that (e.g.)

$$P\big[r \le R \le r + dr\big] = f_{\mathbf{r}}(r)dr.$$

Then

$$f_{\mathbf{r}}(r) = 4\pi r^2 f(r),$$

and

$$g_{\mathbf{s}}(s) = 4\pi s \int_s^\infty \frac{f(r)dr}{\sqrt{r^2 - s^2}},$$

where

$$f(r) = \int f_0(r, v)\boldsymbol{dv}.$$

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Example
Plummer's Model

Used in Simulations

If

$$f_0(r, v) = \frac{c_0}{b^5}\left[\frac{b}{\sqrt{1 + \frac{1}{3}r^2}} - \frac{1}{2}v^2\right]_+^{\frac{7}{2}},$$

where $x_+ = \max[0, x]$, c_0 is a constant, and b is a parameter for which

$$E(V^2) = \langle V^2 \rangle = \frac{3\pi b}{32},$$

then

$$f_{\mathbf{r}}(r) = \frac{r^2}{\sqrt{3(1 + \frac{1}{3}r^2)^5}}$$

and

$$g_{\mathbf{s}}(s) = \frac{2s}{3(1 + \frac{1}{3}s^2)^2}.$$

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The Photometric Data

From Irwin and Hatzidimitrio

1995 *MNRAS*

Fornax

Table 1: Star Counts From Fornax

r	d	n
1.05	29.33	99
2.1	30.57	308
3.15	31.31	526
	...	
74.49	1.45	686
75.54	1.51	725
76.59	1.49	725
77.64	1.4	691
	...	

Totals: For Fornax, circa 150,000 stars in circa 100 bins.

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Estimating f

Some Details

Binned Data: Let

$$0 = r_0 < r_1 < \cdots < r_m < \infty$$

divide the bins and let

$$N_k = \#\{\text{Stars with } r_{k-1} < r \le r_k\}$$

and

$$N = N_1 + \cdots + N_m,$$

the total number of stars.

Estimating G: Let

$$G_{\mathbf{s}}(s) = P\big[S \le s\big] = \int_0^s g_{\mathbf{s}}(t)dt,$$

and

$$G_{\mathbf{s}}^{\#}(r_k) = \frac{N_1 + \cdots + N_k}{N},$$

where Then $G_{\mathbf{s}}^{\#}(r_k)$ is an unbiased estimator for each k.

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An Approxmation: Approxmate f by a step function, say

$$f(r) = f_k \text{ for } r_{k-1} < r \leq r_k$$

for $k = 1, \cdots, m$. Then, after some calculation,

$$1 - G_{\mathbf{s}}(r_{k-1}) = \sum_{j=k}^{m} a_{jk} f_k.$$

where

$$a_{jk} = \frac{4\pi}{3}\left[\sqrt{(r_k^2 - r_{j-1}^2)^3} - \sqrt{(r_k^2 - r_{j-1}^2)^3}\right].$$

So, $f_1, \cdots, f_m$ may be recovered from $G_{\mathbf{s}}$ as

$$f_j = \frac{1}{a_{jj}}\Big[1 - G_{\mathbf{s}}(r_{j-1}) - \sum_{k=j+1}^{m} a_{jk} f_k\Big].$$

Estimation: Replace $G_{\mathbf{s}}(r_{j-1})$ by $G_{\mathbf{s}}^{\#}(r_{j-1})$

Notes 1. Could do better.

2. Very large samples.

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Counts

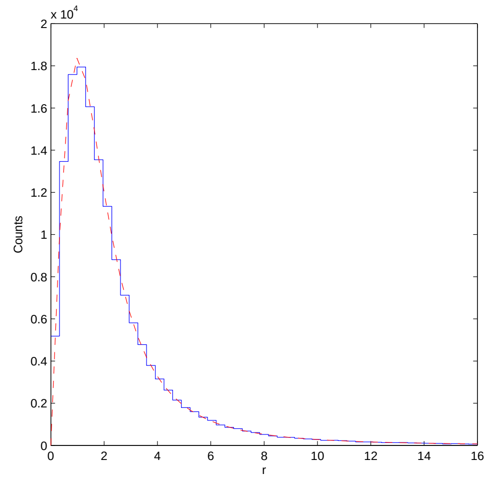

Figure 1: Simulated projected counts for 150,000 stars drawn from a Plummer model.

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Estimated f

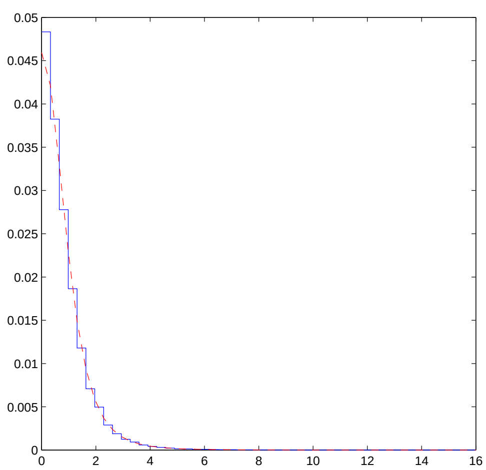

Figure 2: The function f derived from the counts.

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Previews

Estimating μ

Similarities: Let

$$\varphi(r) = f(r)\mu(r) = \int v_3^2 f(r, v)\boldsymbol{dv},$$

like $f(r)$

Differences

- Smaller samples.
- Smoothing (borrow strength).
- Shape restrictions.

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