Stat 426 : Homework 1.

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**Announcement:** The homework carries a total of 45 points, but contributes 5 points towards your total grade.

- **1.** Prove that for three not necessarily disjoint events $A$, $B$ and $C$,

\[ P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C). \]

**Hint:** You can write $A \cup B \cup C$ as $(A \cup B) \cup C$ and use the formula for the union of two events (on page 2 of the first handout) and proceed from there. (5 points)

- **2.** We call $X$ a geometric random variable if $X$ takes values $\{1, 2, 3, \ldots\}$ and $P(X = m) = pq^{m-1}$, where $0 < p, q < 1$ and also $p + q = 1$. Refer to the handout for a random experiment that produces a geometric random variable.

(a) Prove that for any two positive integers $m, n$, it is the case that,

\[ P(X > m + n \mid X > m) = P(X > n). \]

This is the **memoryless property** which is discussed a bit in the Probability Refresher notes. To show this, first prove that the memoryless property is equivalent to the assertion that

\[ P(X > m + n) = P(X > m) P(X > n). \]

Next, show that for the geometric distribution, for any positive integer $l$,

\[ P(X > l) = q^l, \]

and proceed.

(b) We will prove the converse of (a). We will show that if $X$ is a discrete random variable taking values $\{1, 2, 3, \ldots\}$ with probabilities $\{p_1, p_2, p_3, \ldots\}$ and satisfies the memoryless property, then $X$ must follow a geometric distribution.
Follow these steps to establish the fact that $X$ is geometric. Using the fact that $X$ has the memoryless property, show that

$$P(X > m) = (P(X > 1))^m,$$

for any $m \geq 2$. As a first step towards proving this show that

$$P(X > 2) = (P(X > 1))^2.$$ 

Define $p = P(X = 1)$ and $q = P(X > 1)$. You now have,

$$P(X > m) = q^m,$$

for any $m \geq 2$. Use this to show that for any $m \geq 2$,

$$P(X = m) = p q^{m-1}.$$ 

**Hint:** Note that the event $\{X > m - 1\}$ is the disjoint union of the events $\{X > m\}$ and $\{X = m\}$.

But for $m = 1$,

$$P(X = m) = P(X = 1) = p = p q^{m-1},$$

trivially and the proof is complete. (5 + 5 = 10 points)

- 3. If $X$ is random variable with distribution function $F$, with continuous non-vanishing density $f$, obtain the density function of the random variable $Y = X^2$, from first principles; i.e. **without** using the extended change of variable theorem on Page 14 of the first handout.

**Hint:** Express the probability of the event $(X^2 \leq y)$ in terms of the distribution function $F$ of $X$ and proceed from there. (5 points)

- 4. (i) If $X$ and $Y$ are independent standard normal variables find the probability of the event $\{X^2 + Y^2 \leq 1\}$.

(ii) Let $T$ be an exponential random variable with parameter $\lambda$ and let $W$ be a random variable independent of $T$ which assumes the value 1 with probability 1/2 and the value $-1$ with probability 1/2. Show that the density of $X = W T$ is,

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda|x|},$$

using first principles. This distribution is called the **double exponential** distribution.

**Hint:** It would help to split up the event $\{X \leq x\}$ as the union of $\{X \leq x, W = 1\}$ and $\{X \leq x, W = -1\}$. (5 + 5 = 10 points)

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5. Consider a population of $N$ voters who will either vote for Democrats or Republicans (i.e. no one abstains from voting or cancels their vote). The goal is to estimate the proportion $p$ of Democrats in the population.

**Sampling without replacement:** A sample of size $n$ is collected at random without replacement from the population. Let $X_i$ denote the affiliation (1 if Democrat, 0 if Republican) of the $i$th individual in the sample.

(i) Write down the p.m.f. of $X_1$ and the joint p.m.f. of $(X_1, X_2)$. Compute the p.m.f of $X_2$ and show that it is identical to the p.m.f. of $X_1$. Also show that $X_1$ and $X_2$ are not independent.

In general, $X_1, X_2, \ldots, X_n$ all have identical marginal distributions but are dependent. Also, the joint p.m.f. of $(X_i, X_j)$ is the same for all pairs $(i, j)$.

(ii) Consider the special case when the sample size $n$ is equal to the population size $N$, so that your random sample is $(X_1, X_2, \ldots, X_N)$. Let $S = X_1 + X_2 + \ldots + X_N$. Compute $E(S)$ and $\text{Var}(S)$.

(iii) Let $\hat{p} = n^{-1}(X_1 + X_2 + \ldots X_n)$ be the natural estimate of $p$. Compute $E(\hat{p})$ and $\text{Var}(\hat{p})$. (15 points)