Stat 426 : Homework 2.

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February 13, 2004

Announcement: For purposes of the homework, you can cite any results in the handouts or the text-book or any others proved in class, without proof. The homework carries a total of 50 points.

(1) . A sequence of random variables \( \{X_n\} \) is said to converge to a random variable \( X \) in probability if, for every \( \epsilon > 0 \),
\[
P ( | X_n - X | > \epsilon ) \to 0,
\]
as \( n \to \infty \). In particular \( X \) can be a constant, in which case, the sequence of random variables converges in probability to a constant. Now, consider the following set-up: Let \( X_1, X_2, \ldots \) be an infinite sequence of independent random variables each with mean \( \mu \); let \( \sigma_i^2 \) be the variance of \( X_i \) and further assume that the variances are uniformly bounded - in other words, there exists \( K > 0 \) such that \( \sigma_i^2 < K \) for all \( i \). Prove that \( \overline{X}_n = (1/n) (X_1 + X_2 + \ldots + X_n) \) converges in probability to \( \mu \).

Hint: Use Chebyshev’s inequality. (8 points)

Solution: We have,
\[
P ( | X_n - X | > \epsilon ) \leq \frac{\text{Var}(\overline{X}_n)}{\epsilon^2} \leq \frac{\text{Var}(\sum_{i=1}^n X_i)}{n^2 \epsilon^2} = \frac{\sum_{i=1}^n \sigma_i^2}{n^2 \epsilon^2} \leq \frac{nK}{n^2 \epsilon^2} = \frac{K}{n \epsilon^2} \to 0,
\]
as \( n \to \infty \). (In the above derivation we have used the assumption of independence of the \( X_i \)'s and the fact that the variances are uniformly bounded.) It follows that

\[
P( | X_n - X_\cdot | > \epsilon ) \to 0
\]

for any \( \epsilon > 0 \) and this finishes the proof.

(2) (i) For \( \alpha > 0 \), the gamma function is defined as

\[
\Gamma(\alpha) = \int_0^\infty e^{-x} x^{\alpha-1} \, dx.
\]

Show that

\[
\Gamma(\alpha + 1) = \alpha \Gamma(\alpha).
\]

**Hint:** Use integration by parts.

**Solution:** We have,

\[
\Gamma_{\alpha+1} = \int_0^\infty x^{\alpha+1-1} e^{-x} \, dx
\]

\[
= \int_0^\infty x^{\alpha+1-1} e^{-x} \, dx = -x^{\alpha+1-1} e^{-x} \bigg|_0^\infty + \alpha \int_0^\infty x^{\alpha-1} e^{-x} \, dx
\]

\[
= 0 + \alpha \Gamma(\alpha),
\]

where the first term is 0 because \( x^\alpha e^{-x} \) goes to 0 as \( x \) tends to 0 and also as \( x \) tends to \( \infty \).

(ii) Deduce that \( \Gamma(n) = (n - 1)! \).

**Solution:**

\[
\Gamma(n) = (n - 1) \Gamma(n - 1) = (n - 1)(n - 2) \Gamma(n - 2) = \ldots = (n - 1)(n - 2) \ldots 1 \Gamma(1) = (n - 1)!,
\]

where the last step follows on noting that Gamma(1) = 1.

(iii) If \( X_1, X_2, X_3, \ldots \) is a sequence of mutually independent random variables, where each is distributed as exponential with parameter \( \lambda \), then show that the partial sums \( S_n = X_1 + X_2 + \ldots + X_n \) are distributed as Gamma \((n, \lambda)\); in other words, show that the distribution of \( S_n \) is,

\[
f_n(x) = \frac{\lambda^n}{\Gamma(n)} e^{-\lambda x} x^{n-1}.
\]

**Hint:** One way to do this is to prove the result for \( n = 2 \) using the formula for the density of the sum of two independent random variables, as derived in the book (or from first principles). Then, use induction. Assume that the result holds true for \( n - 1 \) and then write \( S_n = S_{n-1} + X_n \) and derive the density of \( S_n \) using the density of \( S_{n-1} \) and \( X_n \) and the fact
that $S_{n-1}$ and $X_n$ are independent.

**Solution:** We show only the induction step, skipping the proof for $n = 2$, which can be viewed as a special case of the induction step. Now, the density of $S_{n-1}$, under the induction hypothesis is,

$$f_{S_{n-1}}(x) = \frac{\lambda^{n-1}}{\Gamma(n-1)} e^{\lambda x} x^{n-2}, x > 0$$

and 0 otherwise, while the density of $X_n$ is,

$$f_{X_n}(y) = \lambda e^{-\lambda y}, y > 0$$

and 0 otherwise. Using the formula on page 94 of Rice, which gives the formula for the density of the sum of two independent random variables, we get, that the density of $S_n = S_{n-1} + X_n$ is,

$$f_{S_n}(z) = \int_0^z \frac{\lambda^{n-1}}{\Gamma(n-1)} e^{-\lambda x} x^{n-2} \lambda e^{-\lambda (z-x)} dx$$

$$= \int_0^z \frac{\lambda^n}{\Gamma(n-1)} e^{-\lambda z} x^{n-2} dx$$

$$= \frac{\lambda^n}{\Gamma(n-1)} e^{-\lambda z} \frac{z^{n-1}}{n-1}$$

$$= \frac{\lambda^n}{\Gamma(n)} e^{-\lambda z} z^{n-1},$$

which is all that is required to show.

- 3. (i) A fair coin is tossed $n$ times and the number of heads $N$ is counted. The coin is then tossed $N$ more times. Find the expected total number of heads generated by this process and the variance.

**Solution:** Let $H$ denote the number of tosses generated in the second phase of this process; i.e. the number of heads out of the $N$ tosses (after the initial $n$ tosses). We know that $N$ follows Binomial($n$, $1/2$) and also that the conditional distribution of $H$ given $N$ is Binomial($N$, $1/2$). To find $E(N + H)$ we proceed as follows:

$$E(N + H) = EN + EH = n/2 + E(E(H \mid N)) = n/2 + E(N/2) = n/2 + n/4 = 3n/4.$$ 

Now $\text{Var}(N + H)$ can be computed as follows:

$$\text{Var}(N + H) = E[\text{Var}(N + H) \mid N] + \text{Var}(E(N + H \mid N)).$$

Now,

$$\text{Var}(N + H) \mid N = \text{Var}(H \mid N) = N/4,$$

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and
\[ E(N + H \mid H) = \frac{3N}{2}. \]

Thus we need to compute,
\[ E(N/4) + \text{Var}(3N/2) = \frac{n}{8} + \frac{9}{4} \times \frac{n}{4} = \frac{11n}{16}. \]

(ii) Let \( Y \) have a density which is symmetric about 0 and let \( X = SY \) where \( S \) is independent of \( Y \) and assumes values 1 and \(-1\) with probability \(1/2\). Show that \( \text{Cov}(X, Y) = 0 \) but that \( X \) and \( Y \) are not independent. (This shows that uncorrelatedness does not necessarily imply independence.)

**Solution:** We know that \( EY = 0 \) because the density of \( Y \) is symmetric about 0. To show that \( \text{Cov}(X, Y) = 0 \) we only need to show that \( E(XY) = 0 \). Now,
\[ E(XY) = E(SY^2) = E(S) \times E(Y^2) = 0 \]
since \( ES = (1/2) \times 1 + (1/2) \times (-1) = 0 \). To show that \( X \) and \( Y \) are not independent, note that, given \( Y = y \), the conditional distribution of \( X = SY \) is concentrated on the set \( \{y, -y\} \); conditionally on \( Y = y \), \( X \) assumes the value \( y \) with probability \(1/2\) and the value \(-y\) with probability \(1/2\). This is clearly different from the unconditional distribution of \( X \) which is continuous and in particular has a density.

(4) (i) A drunkard is standing under a lamp post at time \( t = 0 \). At time \( t = 1 \) he takes a random step to the right with probability 0.6 or a random step to the left with probability 0.4. Then at time \( t = 2 \) he again moves a random step to the right with probability 0.6 or to the left with probability 0.4. He continues in this way. Compute the approximate chance that at time \( t = 100 \) he is within 20 steps of where he started.

**Solution:** Let \( X_1, X_2, \ldots, X_n \) be i.i.d. random variables with \( P(X_1 = 1) = 0.6 \) and \( P(X_1 = -1) = 0.4 \). Here \( X_i \) describes the drunkard’s movement at the \( i \)’th step – 1 corresponds to a step to the right and -1 corresponds to a step to the left. Then \( S = X_1 + X_2 + \ldots + X_{100} \) denotes the drunkard’s displacement and we are required to find \( P(-20 \leq S \leq 20) \). Note that by the CLT, \( S \) is approximately normal with mean \( \mu_n = 100 \times E(X_1) = 20 \) and variance \( \sigma_n^2 = 100 \times \text{variance}(X_1) = 96 \). Now the required probability is equal to
\[
P \left( \frac{-20 - 20}{\sqrt{96}} \leq \frac{S - 20}{\sqrt{96}} \leq 0 \right) = P(-4.08 \leq Z \leq 0)
\]
where \( Z \) follows \( N(0, 1) \). This is approximately 0.5.

(ii) Consider the following game: A fair coin is tossed 900 times and the number of Tails (unknown to you) is recorded. You are then asked to pick 31 numbers. You win $100 if the number of Tails is one of the numbers that you have chosen. Which 31 numbers
should you pick and (then) what is your approximate chance of winning 100 dollars? Explain carefully. (7 + 7 = 14 points)

**Solution:** Let $X_1, X_2, \ldots, X_{900}$ be i.i.d. random variables with $P(X_1 = 1) = P(X_1 = 0) = 0.5$. Here $X_i$ is the outcome on the $i$'th toss - 1 if tails and 0 if heads. Thus $S = X_1 + X_2 + \ldots + X_{900}$ is the total number of tails in 900 tosses and by the CLT is approximately normal with mean $\mu_n = 450$ and variance $\sigma^2_n = 225$. The s.d. therefore is $\sigma_n = 15$.

It is not too difficult to see that the 31 most likely numbers to pick are 435 through 465 (15 numbers on either side of 450). We need to find

$$P(435 \leq S \leq 465) = P\left(\frac{435 - 450}{15} \leq \frac{S - 450}{15} \leq \frac{465 - 450}{15}\right)$$

$$= P(-1 \leq Z \leq 1)$$

$$= 68\% \text{(aprx.)}.$$