Stat 426 : Homework 3 solutions.

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• 2. (i) We know that

\[ \bar{X} \sim N \left( \mu_X, \frac{\sigma_X^2}{n} \right) \text{ and } \bar{Y} \sim N \left( \mu_Y, \frac{\sigma_Y^2}{m} \right), \]

and furthermore \( \bar{X} \) and \( \bar{Y} \) are independent. It follows that

\[ \bar{X} - \bar{Y} \sim N \left( \mu_X - \mu_Y, \frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m} \right), \]

whence,

\[ \frac{\bar{X} - \bar{Y} - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \sim N(0,1). \]

Thus a level \( 1 - \alpha \) confidence interval for \( \mu_X - \mu_Y \) is,

\[ \left\{ \mu_X - \mu_Y : q_{1-\beta_1}^{N(0,1)} \leq \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}}} \leq q_{1-\beta_2}^{N(0,1)} \right\}, \]

where \( \beta_1 + \beta_2 = 1 - \alpha \). This can be written as:

\[ \left[ \bar{X} - \bar{Y} - q_{1-\beta_2}^{N(0,1)} \sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}} \leq \mu_X - \mu_Y \leq \bar{X} - \bar{Y} - q_{1-\beta_1}^{N(0,1)} \sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}} \right]. \]

Choosing \( \beta_1 = \beta_2 = \alpha/2 \) and letting \( z_{\alpha/2} \) denote the \( 1 - \alpha/2 \)’th quantile of the \( N(0,1) \) distribution (and using the symmetry of the normal curve around 0), we obtain a confidence interval of the form,

\[ \left[ \bar{X} - \bar{Y} - \sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}} z_{\alpha/2} \leq \mu_X - \mu_Y \leq \bar{X} - \bar{Y} + \sqrt{\frac{\sigma_X^2}{n} + \frac{\sigma_Y^2}{m}} z_{\alpha/2} \right]. \]

(ii) Note that,

\[ (n - 1) \frac{s_X^2}{\sigma_X^2} = \frac{1}{\sigma_X^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \]
follows a $\chi^2_{n-1}$ distribution, independently of

$$(m - 1) \frac{s_X^2}{\sigma_Y^2} = \frac{1}{\sigma_Y^2} \sum_{i=1}^{m} (Y_i - \bar{Y})^2$$

which follows a $\chi^2_{m-1}$ distribution (see for example Chapter 6 of Rice, Theorem B). It follows that

$$R = \frac{((n - 1) s_X^2 / \sigma_X^2) / (n - 1)}{((m - 1) s_Y^2 / \sigma_Y^2) / (m - 1)} = \frac{s_X^2}{s_Y^2} \times \frac{\sigma_Y^2}{\sigma_X^2} \sim F_{n-1,m-1}.$$ 

With $\beta_1$ and $\beta_2$ as before, a level $1 - \alpha$ confidence interval for $\sigma_X^2 / \sigma_Y^2$ is obtained as,

$$\left\{ \frac{\sigma_X^2}{\sigma_Y^2} : q_{\beta_1}^{F_{n-1,m-1}} \leq \frac{s_X^2 / s_Y^2}{\sigma_X^2 / \sigma_Y^2} \leq q_{1-\beta_2}^{F_{n-1,m-1}} \right\}.$$ 

This can be rewritten as,

$$\frac{s_X^2 / s_Y^2}{q_{1-\beta_2}^{F_{n-1,m-1}}} \leq \frac{\sigma_X^2}{\sigma_Y^2} \leq \frac{s_X^2 / s_Y^2}{q_{\beta_1}^{F_{n-1,m-1}}}.$$ 

(iii) When $\sigma_X^2 = \sigma_Y^2 = \sigma^2$,

$$\frac{(n - 1) s_X^2}{\sigma^2} \sim \chi^2_{n-1} \text{ and } \frac{(m - 1) s_Y^2}{\sigma^2} \sim \chi^2_{m-1};$$

the two random variables in the above display are also independent. Thus,

$$\frac{(n - 1) s_X^2 + (m - 1) s_Y^2}{\sigma^2} \sim \chi^2_{m+n-2}.$$ 

With $\beta_1$ and $\beta_2$ as before, a level $1 - \alpha$ confidence interval for $\sigma^2$ is given by,

$$\left\{ \sigma^2 : q_{\beta_1}^{\chi^2_{m+n-2}} \leq \frac{(n - 1) s_X^2 + (m - 1) s_Y^2}{\sigma^2} \leq q_{1-\beta_2}^{\chi^2_{m+n-2}} \right\}.$$ 

This can be rewritten as,

$$\left[ \frac{(n - 1) s_X^2 + (m - 1) s_Y^2}{q_{1-\beta_2}^{\chi^2_{m+n-2}}} \leq \sigma^2 \leq \frac{(n - 1) s_X^2 + (m - 1) s_Y^2}{q_{\beta_1}^{\chi^2_{m+n-2}}} \right].$$

(iv) When we have the set-up in (iii),

$$\bar{X} - \bar{Y} \sim N \left( \mu_X - \mu_Y, \sigma^2 \left( \frac{1}{n} + \frac{1}{m} \right) \right).$$
Also, note that \( \overline{X} - \overline{Y} \) is independent of \((n-1)s_X^2 + (m-1)s_Y^2\). It follows easily that,

\[
\frac{(\overline{X} - \overline{Y}) - (\mu_X - \mu_Y)}{\sqrt{s_X^2 + (m+n-2)s_Y^2}} \sim t_{m+n-2}.
\]

In other words,

\[
\frac{(\overline{X} - \overline{Y}) - (\mu_X - \mu_Y)}{\sqrt{(n-1)s_X^2 + (m-1)s_Y^2}} \times \sqrt{\frac{(m+n-2)mn}{m+n}} \sim t_{m+n-2}.
\]

Using standard arguments, a level \(1 - \alpha\) confidence interval for \(\mu_X - \mu_Y\) is obtained as,

\[
\left[ \overline{X} - \overline{Y} - \sqrt{\frac{(m+n-2)mn}{m+n}} q_{1-\beta_2}^{t_{m+n-2}}, \overline{X} - \overline{Y} + \sqrt{\frac{(m+n-2)mn}{m+n}} q_{\beta_1}^{t_{m+n-2}} \right].
\]

A level \(\alpha\) test for testing \(\mu_X - \mu_Y = 0\) is obtained as follows: Consider the pivot obtained in (iii) – this follows a \(t_{m+n-2}\) distribution and hence,

\[
P(q_{1-\beta_2}^{t_{m+n-2}} \leq \frac{(\overline{X} - \overline{Y}) - (\mu_X - \mu_Y)}{\sqrt{(n-1)s_X^2 + (m-1)s_Y^2}} \times \sqrt{\frac{(m+n-2)mn}{m+n}} \sim t_{m+n-2} \leq q_{1-\beta_2}^{t_{m+n-2}}) = 1 - \alpha.
\]

Under \(H_0\), the pivot becomes,

\[
P_0 = \frac{(\overline{X} - \overline{Y})}{\sqrt{(n-1)s_X^2 + (m-1)s_Y^2}} \times \sqrt{\frac{(m+n-2)mn}{m+n}}.
\]

Our level \(\alpha\) test \(\phi\) is then given as follows:

\[
\phi(X, Y) = 1 \text{ if } P_0 \notin [q_{\beta_1}^{t_{m+n-2}}, q_{1-\beta_2}^{t_{m+n-2}}]
\]

and is equal to 0 otherwise.

(v) The fact that \(D\) is approximately distributed as \(N(0, 1)\) as \(n, m \to \infty\) is not easy to establish. We assume that this is indeed the case and proceed. It now follows that

\[
\text{Prob} \left( -z_{\alpha/2} \leq \frac{(\overline{X} - \overline{Y}) - (\mu_X - \mu_Y)}{\sqrt{s_X^2/n + s_Y^2/m}} \leq z_{\alpha/2} \right) = \text{approx. } 1 - \alpha.
\]

By the usual algebra resorted to in the previous examples and worked out in class, it follows that the level \(1 - \alpha\) C.I. for \(\mu_X - \mu_Y\) that we obtain is given by

\[
\left[ \overline{X} - \overline{Y} - \sqrt{s_X^2/n + s_Y^2/m} z_{\alpha/2}, \overline{X} - \overline{Y} + \sqrt{s_X^2/n + s_Y^2/m} z_{\alpha/2} \right].
\]
4. We are given that \( X \) and \( Y \) are independent random variables each following \( \text{exp}(\lambda) \). Now,

\[
\frac{X}{Y} = \frac{2\lambda X/2}{2\lambda Y/2}.
\]

It can be shown (either by using the Jacobian Theorem or by directly computing the distribution function) that \( 2\lambda X \) follows \( \text{exp}(1/2) \) which is just \( \chi^2_2 \) and \( 2\lambda Y \) also follows \( \chi^2_2 \) (see the paragraph below). Furthermore, \( 2\lambda X \) and \( 2\lambda Y \) are independent; it follows that

\[
\frac{X}{Y} = \frac{2\lambda X/2}{2\lambda Y/2} \sim F_{2,2}
\]

directly from the definition of an \( F \) random variable.

Finding the distribution of \( 2\lambda X \): Note that \( P(X > x) = e^{-\lambda x} \). So,

\[
P(2\lambda X > w) = P(X > w/2\lambda) = e^{-\lambda w/2\lambda} = e^{-w/2}.
\]

This shows that \( P(2\lambda X \leq w) = 1 - e^{-w/2} \), showing that \( 2\lambda X \) indeed follows Exponential (0.5).

5. Using a derivation analogous to the one above, we can show that \( \lambda X \) follows Exponential(1) – thus, it becomes a pivot. Let \( q_\beta \) denote the \( \beta \)th quantile of the Exponential(1) distribution. Note that the distribution function of Exponential(1) is \( G(x) = 1 - e^{-x} \). Hence we must have: \( G(q_\beta) = 1 - e^{-q_\beta} = \beta \), whence \( q_\beta = -\log(1 - \beta) \). Now, we have:

\[
P(q_{\alpha/2} \leq \lambda X \leq q_{1-\alpha/2}) = 1 - \alpha.
\]

Setting \( \alpha = 0.05 \) and plugging in the formula for \( q_\beta \) we obtain

\[
P \left( \frac{-\log(.05)}{X} \leq \lambda \leq \frac{-\log(.95)}{X} \right) = 0.95.
\]

Thus a 95% C.I. for \( \lambda \) is given by:

\[
\left[ \frac{-\log(.05)}{X}, \frac{-\log(.95)}{X} \right].
\]