Statistics 426 : Homework 5 solutions.

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\textbf{Note:} For the testing problems where $\sigma^2$ is unknown, do not use the Neyman-Pearson lemma, since you are dealing with composite hypotheses both under the null and the alternative. The homework carries a total of 60 points out of which the maximum you can score is 50 and contributes 6.25 points towards your total grade.

(1) Consider \textit{i.i.d.} observations $X = (X_1, X_2, \ldots, X_n)$ from a $N(\mu, \sigma^2)$ distribution. Assume that $\sigma^2$ is known. Consider testing the null hypothesis $H_0 : \mu = \mu_0$ versus $H_1 : \mu = \mu_1$ where $\mu_1 < \mu_0$. We are interested in finding a Most Powerful level $\alpha$ test for this testing problem.

(a) Use the Neyman–Pearson lemma to show that the most powerful level $\alpha$ test can be written as: $\phi_{MP}(X) = 1$ if $\overline{X} \leq c_{\alpha, n}$ and $\phi_{MP}(X) = 0$ otherwise.

\textbf{Solution:} As in the hypothesis testing notes, formulating the problem in the Neyman–Pearson set-up, we have:

\[ f_1(X) = f_{\mu_1}(X_1, X_2, \ldots, X_n) \text{ and } f_0(X) = f_{\mu_0}(X_1, X_2, \ldots, X_n), \]

where

\[ f_{\mu}(X_1, X_2, \ldots, X_n) = \left( \frac{1}{\sqrt{2\pi \sigma}} \right)^n \exp \left( -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu)^2 \right). \]

The Neyman-Pearson test is based on the ratio $f_1(X)/f_0(X)$ or equivalently on $\log f_1(X)/f_0(X)$. Thus, the MP level $\alpha$ test is of the form:

\[ \phi_{MP}(X) = 1 \text{ if } \log \frac{f_1(X)}{f_0(X)} > \log K_\alpha \equiv k_\alpha, \]

\[ \phi_{MP}(X) = \gamma_\alpha \text{ if } \log \frac{f_1(X)}{f_0(X)} = k_\alpha \]

and

\[ \phi_{MP}(X) = 0 \text{ otherwise}. \]
We have,

\[
\log \frac{f_1(X)}{f_0(X)} = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu_1)^2 + \frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu_0)^2
\]

\[= \frac{n}{\sigma^2} (\mu_1 - \mu_0) + \frac{n}{2\sigma^2} (\mu_0^2 - \mu_1^2).\]

Since \(\mu_1 - \mu_0 < 0\), \(\log f_1(X)/f_0(X)\) is a strictly decreasing function of \(X\) and we can base our MP test \(\phi_{MP}\) simply on \(X\). This will assume the value 1 when \(X < c_{a,n}\), will be equal to \(\gamma_\alpha\) when \(X = c_{a,n}\) and equal to 0 otherwise. Also, since \(\phi_{MP}\) is of level \(\alpha\), it needs to satisfy \(E_0(\phi_{MP}(X)) = \alpha\) or equivalently,

\[P_0(X < c_{a,n}) + \gamma_\alpha P_0(X = c_{a,n}) = \alpha.\]

Since \(X\) follows \(N(\mu_0, \sigma^2/n)\) under \(H_0\), the chance that \(X = c_{a,n}\) is actually 0 and we can choose our test function to be a 0-1 valued test function, with \(\gamma_\alpha = 1\), so that \(\phi(X) = 1\) if \(X \leq c_{a,n}\) (the rejection region of the test) and \(\phi(X) = 0\) otherwise (the acceptance region of the test). We then have,

\[P_0(X \leq c_{a,n}) = \alpha,\]

or equivalently,

\[P_0 \left( \frac{\sqrt{n}(X - \mu_0)}{\sigma} \leq \frac{\sqrt{n}(c_{a,n} - \mu_0)}{\sigma} \right) = \alpha.\]

Under \(H_0\),

\[\frac{\sqrt{n}(X - \mu_0)}{\sigma} \sim N(0,1)\]

showing that

\[\frac{\sqrt{n}(c_{a,n} - \mu_0)}{\sigma} = z_\alpha = -z_{1-\alpha}.\]

Thus, our MP test can be formulated as: Reject \(H_0\) in favor of \(H_1\) if

\[X \leq \mu_0 - \frac{\sigma}{\sqrt{n}} z_{1-\alpha}\]

and do not reject otherwise.

(b) Using the fact that the probability of Type 1 error for the most powerful test level \(\alpha\) test must actually be equal to \(\alpha\), conclude that

\[c_{a,n} = \mu_0 - \frac{\sigma}{\sqrt{n}} z_{1-\alpha}.\]

The solution to (b) has been incorporated in the solution to (a).

(c) Consider the power function of the test

\[\beta(\mu) = \text{Prob}_\mu \left( X \leq \mu_0 - \frac{\sigma}{\sqrt{n}} z_{1-\alpha} \right).\]
Show that the power function is a decreasing function of $\mu$ and that $\beta(\mu_0) = \alpha$.

**Solution:** We have,

$$
\beta(\mu) = E_\mu(\phi(X)) = P_\mu \left( X \leq \mu_0 - \frac{\sigma}{\sqrt{n}} z_{1-\alpha} \right) = P_\mu \left( \frac{\sqrt{n}(X - \mu)}{\sigma} \leq -z_{1-\alpha} \right) = P_\mu \left( \frac{\sqrt{n}(X - \mu)}{\sigma} + \frac{\sqrt{n}(\mu - \mu_0)}{\sigma} \leq -z_{1-\alpha} \right).
$$

Under $\mu$ i.e. when the underlying distribution is $N(\mu, \sigma^2)$, $\sqrt{n}(X - \mu)/\sigma \sim N(0, 1)$; if $Z$ denotes a standard normal random variable we get,

$$
\beta(\mu) = P \left( Z + \frac{\sqrt{n}(\mu - \mu_0)}{\sigma} \leq -z_{1-\alpha} \right) = \Phi \left( -z_{1-\alpha} - \frac{\sqrt{n}(\mu - \mu_0)}{\sigma} \right) = 1 - \Phi \left( z_{1-\alpha} + \frac{\sqrt{n}(\mu - \mu_0)}{\sigma} \right).
$$

Since $\Phi \left( z_{1-\alpha} + \frac{\sqrt{n}(\mu - \mu_0)}{\sigma} \right)$ is an increasing function of $\mu$ it follows that $\beta(\mu)$ is a decreasing function of $\mu$. That $\beta(\mu_0)$ is equal to $\alpha$ is checked by direct verification. As $\mu$ increases from $-\infty$ to $\infty$ $\beta(\mu)$ decreases from 1 to 0.

(d) **Remark:** Since the most powerful test $\phi_{MP}$ for testing the null hypothesis $\mu = \mu_0$ versus the alternative $\mu = \mu_1$ does not depend on the alternative $\mu_1$, it follows that $\phi_{MP}$ is also MP level $\alpha$ for testing $\mu = \mu_0$ versus $\mu < \mu_0$. Furthermore, as the power function decreases with $\mu$, $\beta(\mu) \leq \beta(\mu_0) = \alpha$ for $\mu > \mu_0$. Thus, $\phi_{MP}$ is also MP level $\alpha$ for testing $H_0 : \mu \geq \mu_0$ versus $H_1 : \mu < \mu_0$. I don’t need you to do anything for part (d), but do convince yourselves that this is indeed the case. (6 + 4 + 5 = 15 points)

(2) Consider testing $H_0 : \mu = \mu_0$ versus $H_1 : \mu \neq \mu_0$ based on a sample $X = (X_1, X_2, \ldots, X_n)$ from a $N(\mu, \sigma^2)$ distribution, but now assume that $\sigma^2$ is unknown. The above testing problem was worked out in the notes when $\sigma^2$ is known and we used the $z$-statistic,

$$
z = \frac{\sqrt{n}(X - \mu_0)}{\sigma}.
$$

When $\sigma$ is unknown, a natural thing to do is to consider the version of the $z$ statistic with $\sigma$ replaced by $s$ where $s^2$ is the usual unbiased estimate of $\sigma^2$ (see, for example, Page 6 of the
Hypothesis Testing notes). Consider the test,

\[ \phi^*(X) = 1 \text{ if } |T| = \left| \frac{\sqrt{n} (\bar{X} - \mu_0)}{s} \right| > \kappa \]

and

\[ \phi^*(X) = 0 \text{ otherwise}, \]

where \( \kappa \) is some positive constant.

**Notation:** In what follows, the \( t_k \) distribution function will be denoted by \( F_k \) and the \( t_k \) density will be denoted by \( f_k \). We will use the fact that \( f_k(x) = f_k(-x) \) and that \( f_k \) decreases as \( x \) increases from 0 to \( \infty \) (and consequently also decreases as \( x \) decreases from 0 to \( -\infty \)). Also, \( F_k(-x) = 1 - F_k(x) \).

(a) Is this a reasonable test statistic for the testing problem? Explain your answer.

**Solution:** This is reasonable, since the null hypothesis gets rejected when the absolute deviation of the sample mean from the population mean under the null hypothesis is large compared to the estimated variability in the population. Since the alternative hypothesis lies on either side of \( \mu_0 \) it is necessary to consider large deviations on either side of \( \mu_0 \) as being significant.

(b) If we choose \( \kappa = t_{n-1:1-\alpha/2} \), where \( t_{n-1:1-\alpha/2} \) is the \( 1 - \alpha/2 \)th quantile of the \( t \) distribution on \( n - 1 \) degrees of freedom, show that the probability of Type 1 error of \( \phi^* \) is exactly equal to \( \alpha \).

**Solution:** We have,

\[ E_{\mu_0}(\phi^*(X)) = P_{\mu_0} \left( |T| = \left| \frac{\sqrt{n} (\bar{X} - \mu_0)}{s} \right| > t_{n-1:1-\alpha/2} \right). \]

Now, under \( H_0 \),

\[ \frac{\sqrt{n} (\bar{X} - \mu_0)}{s} = \frac{\sqrt{n} (\bar{X} - \mu_0)/\sigma}{\sqrt{s^2/\sigma^2}} \],

follows a \( t_{n-1} \) distribution, since

\[ \frac{\sqrt{n} (\bar{X} - \mu_0)}{\sigma} \sim N(0,1) \]

and

\[ \frac{s^2}{\sigma^2} = \frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2 / \sigma^2 \sim \chi^2_{n-1} / (n-1) \]

and \( \bar{X} \) and \( s^2 \) are independent. Thus, the probability of Type 1 error is,

\[ E_{\mu_0}(\phi^*(X)) = P \left( |V| \geq t_{n-1:1-\alpha/2} \right) \]
where $V$ follows a $t_{n-1}$ distribution. This is simply,

$$P(V \geq t_{n-1;1-\alpha/2}) + P(V \leq -t_{n-1;1-\alpha/2}) = 1 - F_{n-1}(t_{n-1;1-\alpha/2}) + F_{n-1}(-t_{n-1;1-\alpha/2}) = \alpha/2 + \alpha/2 = \alpha.$$  

(c) Let $\beta(\mu)$ as before denote the power function of the above test. Thus,

$$\beta(\mu) = \text{Prob}_\mu (\phi^*(X) = 1) = \text{Prob}_\mu \left( \left| \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} \right| > t_{n-1;1-\alpha/2} \right).$$

Show that $\beta(\mu_0) = \alpha$, and by using steps similar to the derivation on pages 4 through 5 of the Hypothesis Testing notes show that $\beta(\mu)$ hits a minimum at $\mu_0$, is symmetric around $\mu_0$ (i.e. $\beta(\mu_0 + \Delta) = \beta(\mu_0 - \Delta)$) and tends to 1 as $\mu$ tends to $\infty$ or $-\infty$. Also, plot (schematically) the power function.

**Solution:** We have

$$\beta(\mu) = P_{\mu} \left( \left| \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} \right| > t_{n-1;1-\alpha/2} \right).$$

For the moment, let’s omit this part, as this is more difficult than I had thought. I will include the derivation later but it is not important for the purpose of the midterm or the final.

(d) Consider now the testing problem $H_0 : \mu \leq \mu_0$ versus $H_1 : \mu > \mu_0$ but with the same data as before and $\sigma^2$ still unknown. Consider the test function, $\phi(X)$ defined as follows:

$$\phi(\bar{X}) = 1 \text{ if } \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} > t_{n-1;1-\alpha}$$

and $\phi(X) = 0$ otherwise.

(i) Is this a reasonable test statistic for this testing problem? Explain your answer.

**Solution:** This is a reasonable test statistic for the testing problem; it rejects when the sample mean deviates substantially to the right of the population mean (under the null hypothesis) relative to the underlying variability. This makes sense because under the alternative the mean is stipulated to be larger than $\mu_0$. Deviations to the left of $\mu_0$ are not incompatible with the null hypothesis.

(ii) Let

$$\bar{\beta}(\mu) = \text{Prob}_\mu \left( \phi(\bar{X}) = 1 \right)$$

denote the power function of this test. Show that $\bar{\beta}(\mu)$ is increasing in $\mu$, that $\bar{\beta}(\mu_0) = \alpha$ and that $\beta(\mu)$ converges to 1 as $\mu$ tends to infinity. Also, plot the power function schematically.

**Solution:** Once again, a rigorous proof is omitted for the moment. The issues
involved are the same as with the previous problem. It is easy to show however that
\[ \beta(\mu_0) = \alpha; \text{ under } \mu = \mu_0, \]
\[ \frac{\sqrt{n}(\bar{X} - \mu_0)}{s} \sim t_{n-1} \]
so that
\[ \beta(\mu_0) = P(T > t_{n-1; 1-\alpha}), \]
where \( T \) follows a \( t_{n-1} \) distribution. Thus,
\[ \beta(\mu_0) = 1 - F_{n-1}(t_{n-1; 1-\alpha}) = 1 - (1 - \alpha) = \alpha. \]

(iii) Conclude that \( \tilde{\phi} \) is a level \( \alpha \) test. (3 + 4 + 8 + (3 + 6 + 3) = 27 points)

**Solution:** This follows from the fact that \( \beta(\mu) \) increases with \( \mu \). Thus, for \( \mu < \mu_0, \beta(\mu) \leq \beta(\mu_0) = \alpha. \) This shows that \( \tilde{\phi} \) is level \( \alpha \).

(3) The Two Sample Problem: Consider two samples from two populations. The first sample is \( X = (X_1, X_2, \ldots, X_n) \) from \( N(\mu_X, \sigma_X^2) \) and the second sample is \( Y = (Y_1, Y_2, \ldots, Y_m) \) from \( N(\mu_Y, \sigma_Y^2) \). The two samples can be assumed to be independent.

We discussed a situation of this sort in class. Think of the first sample as measured blood pressures of a group of individuals with high blood pressure who have been on placebo for a month and the second sample as measured blood pressures of a group of individuals with high blood pressure who have been on some pressure drug that claims to work. The individuals were randomly assigned to the treatment group or the placebo group at the beginning of the clinical trial. If we are not willing to introduce the new drug unless we are sure that it does a really good job (we might take a conservative stand if we know that the drug has some undesirable side effects) we could formulate the null hypothesis as \( H_0 : \mu_X \leq \mu_Y \equiv \mu_X - \mu_Y \leq 0 \) versus \( H_1 : \mu_X > \mu_Y \equiv \mu_X - \mu_Y > 0 \).

A reasonable test statistic for this problem could be based on the difference of the sample means \( \bar{X} - \bar{Y} \) which is an unbiased estimator of \( \mu_X - \mu_Y \). Large values of this statistic are more compatible with the alternative hypothesis. Let \( s_X^2, s_Y^2 \) be the usual unbiased estimator of \( \sigma_X^2 \) and \( s_Y^2 \) denote the usual unbiased estimator of \( \sigma_Y^2 \).

(i) Consider first the situation where you know \( \sigma_X^2 \) and \( \sigma_Y^2 \). From Homework 4 you know that
\[ \bar{X} - \bar{Y} \sim N(\mu_X - \mu_Y, \sigma_X^2/n + \sigma_Y^2/m). \]

Use the above fact, along with the ideas we came across while developing a test for \( \mu \leq \mu_0 \) versus \( \mu > \mu_0 \) for an i.i.d. sample from a normal distribution with known variance, to construct a level \( \alpha \) test for the above problem that “accepts” or rejects based on the value of \( \bar{X} - \bar{Y} \). Justify your steps carefully.
Solution: Let

$$\sigma_{comb}^2 = \sigma_X^2/n + \sigma_Y^2/m$$

and

$$\mu_{diff} = \mu_X - \mu_Y.$$  

We know that,

$$\frac{(\bar{X} - \bar{Y}) - \mu_{diff}}{\sigma_{comb}} \sim N(0, 1).$$

In our setting $H_0 : \mu_{diff} \leq \mu_0 \equiv 0$ and $H_1 : \mu_{diff} > 0$. A level $\alpha$ test that “accepts” or rejects based on the value of $\bar{X} - \bar{Y}$ is given by

$$\phi(X, Y) = 1 \text{ if } \frac{(\bar{X} - \bar{Y}) - \mu_0}{\sigma_{comb}} \geq c_\alpha$$

and $\phi(X, Y) = 0$ otherwise. The power function of this test is,

$$\beta(\mu_{diff}) = P_{\mu_{diff}} \left( \frac{(\bar{X} - \bar{Y}) - \mu_0}{\sigma_{comb}} \geq c_\alpha \right).$$

Choosing $c_\alpha = z_{1-\alpha}$ gives us a level $\alpha$ test, since we then have,

$$\beta(\mu_{diff}) = P_{\mu_{diff}} \left( \frac{(\bar{X} - \bar{Y})}{\sigma_{comb}} \geq z_{1-\alpha} \right).$$

When $\mu_{diff} = \mu_0 = 0$, we have

$$\frac{(\bar{X} - \bar{Y})}{\sigma_{comb}} \sim N(0, 1);$$

therefore,

$$\beta(0) = P(Z \geq z_{1-\alpha}) = \alpha.$$  

That the power function is increasing in $\mu_{diff}$ follows on noting that

$$\beta(\mu_{diff}) = P_{\mu} \left( \frac{(\bar{X} - \bar{Y}) - \mu_{diff}}{\sigma_{comb}} + \frac{\mu_{diff}}{\sigma_{comb}} \geq z_{1-\alpha} \right) = P \left( Z + \frac{\mu_{diff}}{\sigma_{comb}} \geq z_{1-\alpha} \right);$$

this is easily checked to be an increasing function of $\mu_{diff}$ following the derivation on pages 10–11 of the hypothesis testing notes. Thus $\beta(\mu_{diff}) \leq 0$ for $\mu_{diff} < 0$ showing that $\phi(X, Y)$ is level $\alpha$.

(ii) Now suppose that you do not know the underlying variances but that you do know $\sigma_X^2 = \sigma_Y^2 = \sigma^2$. In this case, show that

$$\frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\hat{\sigma} \sqrt{1/m + 1/n}} \sim t_{n-1},$$
where
\[ \hat{\sigma} = \sqrt{\frac{\hat{\sigma}^2}{m+n-2}} = \frac{1}{m+n-2} \left( (n-1) s_X^2 + (m-1) s_Y^2 \right). \]

I basically discussed this in class while giving out hints for Homework 4.

**Solution:** Note that, in this situation,
\[ \bar{X} \sim N(\mu_X, \sigma^2/n) \quad \bar{Y} \sim N(\mu_Y, \sigma^2/n) \]

and
\[ (n-1) \frac{s_X^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^{n} (X_i - \bar{X})^2 \sim \chi^2_{n-1} \quad (m-1) \frac{s_Y^2}{\sigma^2} = \frac{1}{\sigma^2} \sum_{i=1}^{m} (Y_i - \bar{Y})^2 \sim \chi^2_{m-1}. \]

Also \( \bar{X}, \bar{Y}, s_X^2, s_Y^2 \) are all mutually independent. Thus, it follows that,
\[ \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{\sigma^2}{m+1/n}}} \sim N(0, 1) \]

and
\[ \frac{(n-1) s_X^2 + (m-1) s_Y^2}{\sigma^2} \sim \chi^2_{m+n-2} \]

independently of \( (\bar{X} - \bar{Y}) \). Thus,
\[ \frac{(\bar{X} - \bar{Y}) - (\mu_X - \mu_Y)}{\sqrt{\frac{1}{m+n-2} (n-1) s_X^2 + (m-1) s_Y^2}} \equiv \frac{(\bar{X} - \bar{Y}) - \mu_{\text{diff}}}{\hat{\sigma} (\sqrt{1/m + 1/n})} \equiv \frac{(\bar{X} - \bar{Y}) - \mu_{\text{diff}}}{\hat{\sigma} (\sqrt{1/m + 1/n})} \sim t_{m+n-2}. \]

(iii) Use the above fact to develop a level \( \alpha \) test for testing \( H_0 \) against \( H_1 \) that rejects for large values of \( \bar{X} - \bar{Y} \). This will give you, what is called a “two sample t test”. ( 6 + 6 + 6 = 18 points)

**Solution:** The test statistic we propose for testing \( \mu_{\text{diff}} \leq 0 \) versus \( \mu_{\text{diff}} > 0 \) is simply,
\[ \phi(X, Y) = 1 \text{ if } \frac{(\bar{X} - \bar{Y})}{\hat{\sigma} (\sqrt{1/m + 1/n})} > t_{m+n-2; 1-\alpha} \]

and is 0 otherwise. Let \( \beta(\cdot) \) denote the power function of the test. This is a function of \( \mu_{\text{diff}} \), the difference of the sample means. It is easy to see that \( \beta(0) = \alpha \). It can also be proved that \( \beta(\cdot) \) is an increasing function of \( \mu_{\text{diff}} \) and runs from 0 to 1 as \( \mu_{\text{diff}} \) runs from \( -\infty \) to \( \infty \). This also shows that \( \phi \) is a level \( \alpha \) test.