Stat 426 : Homework 1.

Moulinath Banerjee

University of Michigan

January 20, 2003

Announcement: For purposes of the homework, you can cite any results in the handouts or the text-book or any others proved in class, without proof. The homework carries a total of 60 points, but contributes 5 points towards your total grade.

1. Prove that for three not necessarily disjoint events $A$, $B$ and $C$,

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(A \cap B) - P(B \cap C) - P(A \cap C) + P(A \cap B \cap C).$$

Hint: You can write $A \cup B \cup C$ as $(A \cup B) \cup C$ and use the formula for the union of two events (on page 2 of the first handout) and proceed from there. (5 points)


3. (a) Show that if events $A_1,A_2,\ldots,A_n$ are mutually independent, then so are $A_1,A_2,\ldots,A_{n-1},A_n^c$. (Hint: Use the definition of mutual independence)

(b) Use this result repeatedly to show that if $B_1,B_2,\ldots,B_n$ are independent events then so are $C_1,C_2,\ldots,C_n$ where each $C_i$ is either $B_i$ or $B_i^c$. (Hint: Observe that it suffices to prove that if $B_1,B_2,\ldots,B_n$ are independent, then so are $B_1,B_2,\ldots,B_m,B_{m+1}^c,B_{m+2}^c,\ldots,B_n^c$. (Why ?)) (3 + 2 = 5 points)

4. We call $X$ a geometric random variable if $X$ takes values $\{1,2,3,\ldots\}$ and $P(X = m) = pq^{m-1}$, where $0 < p,q < 1$ and also $p + q = 1$. Refer to the handout for a random experiment that produces a geometric random variable.

(a) Prove that for any two positive integers $m,n$, it is the case that,

$$P(X > m + n \mid X > m) = P(X > n).$$

This is the memoryless property which is discussed a bit in the Probability Refresher notes. To show this, first prove that the memoryless property is equivalent to the assertion that

$$P(X > m + n) = P(X > m)P(X > n).$$
Next, show that for the geometric distribution, for any positive integer \( l \),
\[
P(X > l) = q^l,
\]
and proceed.

(b) We will prove the converse of (a). We will show that if \( X \) is a discrete random variable taking values \( \{1, 2, 3, \ldots \} \) with probabilities \( \{p_1, p_2, p_3, \ldots \} \) and satisfies the memoryless property, then \( X \) must follow a geometric distribution.

Follow these steps to establish the fact that \( X \) is geometric. Using the fact that \( X \) has the memoryless property, show that
\[
P(X > m) = (P(X > 1))^m,
\]
for any \( m \geq 2 \). As a first step towards proving this show that
\[
P(X > 2) = (P(X > 1))^2.
\]

Define \( p = P(X = 1) \) and \( q = P(X > 1) \). You now have,
\[
P(X > m) = q^m,
\]
for any \( m \geq 2 \). Use this to show that for any \( m \geq 2 \),
\[
P(X = m) = p q^{m-1}.
\]

**Hint:** Note that the event \( \{X > m - 1\} \) is the disjoint union of the events \( \{X > m\} \) and \( \{X = m\} \).

But for \( m = 1 \),
\[
P(X = m) = P(X = 1) = p = p q^{m-1},
\]
trivially and the proof is complete. (5 + 5 = 10 points)

- **5.** If \( X \) is random variable with distribution function \( F \), with continuous non-vanishing density \( f \), obtain the density function of the random variable \( Y = X^2 \), from first principles; i.e. **without** using the extended change of variable theorem on Page 14 of the first handout.

**Hint:** Express the probability of the event \( X^2 \leq y \) in terms of the distribution function \( F \) of \( X \) and proceed from there. (5 points)
6. (i) If $X$ and $Y$ are independent standard normal variables find the probability of the event \{ $X^2 + Y^2 \leq 1$ \}.

(ii) Let $T$ be an exponential random variable with parameter $\lambda$ and let $W$ be a random variable independent of $T$ which assumes the value 1 with probability $1/2$ and the value $-1$ with probability $1/2$. Show that the density of $X = WT$ is,

$$f_X(x) = \frac{\lambda}{2} e^{-\lambda |x|},$$

using first principles. This distribution is called the double exponential distribution.

**Hint:** It would help to split up the event \{ $X \leq x$ \} as the union of \{ $X \leq x, W = 1$ \} and \{ $X \leq x, W = -1$ \}. (5 + 5 = 10 points)

7. (i) If $X$ and $Y$ are independent Poisson random variables with parameters $\lambda_1$ and $\lambda_2$, then show that $X + Y$ is also Poisson with parameter $\lambda_1 + \lambda_2$. Recall that if $W$ follows Poisson($\theta$), then the p.m.f. of $W$ is,

$$P(W = m) = \frac{e^{-\theta} \theta^m}{m!}.$$

**Hint:** Write $P(X + Y = m)$ as $\sum_{i=0}^{m} P(X = i, X + Y = m - i)$ and proceed.

(ii) Use this result repeatedly to show that if $X_1, X_2, \ldots, X_n$ are independent Poisson random variables with parameters $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively, then $X_1 + X_2 + \ldots + X_n$ follows Poisson with parameter $\lambda_1 + \lambda_2 + \ldots + \lambda_n$.

(iii) Show from first principles that the conditional distribution of the random vector $(X_1, X_2, \ldots, X_n)$ given that the sum $X_1 + X_2 + \ldots + X_n = K$ for some integer $K$ follows the multinomial distribution with parameters $(K, p_1, p_2, \ldots, p_n)$ where each $p_i$ is given by $\lambda_i / (\sum_{j=1}^{n} \lambda_j)$.

The above result has an interesting “Poisson process interpretation” which we will discuss if we have time. (3 + 2 + 5 = 10 points)

8. Problem 54 on Page 109 of the text book. (10 points)


I think this is a nice problem because it’s a nice exercise in manipulating conditional probabilities, something indispensable in statistics, and in my experience one that takes a quite a long time, experience and exposure.