The likelihood function $L(\theta)$ can be written as

$$L(\theta | x_1, \ldots, x_n) = \prod_{i=1}^{n} \frac{\theta^2}{2} 1(1 | x_i | \leq \frac{1}{\theta^2})$$

$$= \frac{\theta^{2n}}{2^n} 1(\max | x_i | \leq \frac{1}{\theta^2})$$

Thus, the value $\hat{\theta}_{MLE}$ that maximizes this function is

$$\hat{\theta}_{MLE} = \frac{1}{\sqrt{\max | x_i |}}$$

Also, $|x_i| \sim \text{Unif}(0, \frac{1}{\theta^2})$

Thus, $E[|x_i|] = \frac{1}{2\theta^2}$

i.e. $\theta = \frac{1}{\sqrt{2E|\text{max} | x_i |}}$

Thus, a MOM estimator is

$$\hat{\theta}_{MOM} = \frac{1}{\sqrt{2 \cdot \frac{1}{n} \sum_{i=1}^{n} |x_i|}}$$
2. \[ X_i \sim N(\mu_1, \sigma^2) \]
\[ Y_i \sim N(\mu_2, \sigma^2) \]
\[ \text{cor}(X_i, Y_i) = \rho \]

Thus, \[ X_i - Y_i \sim N(\mu_1 - \mu_2, 2\sigma^2 + 2\sigma^2 \rho) \]

define, \[ \omega^2 = 2\sigma^2 + 2\sigma^2 \rho \]

then, \[ X_i - Y_i \sim N(\mu_1 - \mu_2, \omega^2) \] \[ \text{[\omega^2 unknown]} \]

\[
\frac{1}{\text{Pr}} \left[ - t_{(m)}^{(a/2)} \leq \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\bar{Y}/\sqrt{n}} \leq t_{(m)}^{(1-a/2)} \right] = 1 - \alpha
\]

where \( t_{(m)}^{(a/2)} \) and \( t_{(1-a/2)}^{(m)} \) are quantiles of \( t \) distribution with d.f. = \( n \) and

\[ s^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left( (X_i - Y_i) - (\bar{X} - \bar{Y}) \right)^2 \]

Thus, the required CI is

\[ \left( (\bar{X} - \bar{Y}) \pm t_{(m)}^{(a/2)} \frac{s}{\sqrt{n}} \right) \]
(ii) we have seen that,

\[ T = \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{S/\sqrt{n}} \sim t^{(n)} \]

Now,

\[ P_{1-\alpha} \left( T < t^{(n)}_{1-\alpha} \right) = 1 - \alpha \]

where \( t^{(n)}_{1-\alpha} \) is the \((1-\alpha)\) th quantile of a \( t^{(n)} \) distribution.

Thus, \( P_{1-\alpha} \left( \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{S/\sqrt{n}} < t^{(n)}_{1-\alpha} \right) = 1 - \alpha \)

or, \( P_{1-\alpha} \left( (\bar{X} - \bar{Y}) - t^{(n)}_{1-\alpha} \frac{S}{\sqrt{n}} < (\mu_1 - \mu_2) \right) = 1 - \alpha \)

Thus,

\[ a = (\bar{X} - \bar{Y}) - t^{(n)}_{1-\alpha} \frac{S}{\sqrt{n}} \]

where, as before,

\[ S^2 = \frac{1}{n-1} \sum_{i=1}^{n} \left( X_i - Y_i \right)^2 - (\bar{X} - \bar{Y})^2 \]
(3) The likelihood function,

\[ L(\theta|x_1, \ldots, x_n) = \frac{n}{\theta+1} \prod_{i=1}^{n} x_i^\theta 1_{0 \leq x_i < 1} \prod_{i=1}^{n} 1_{\theta > 0} \]

\[ = (\theta+1)^{-n} \left( \prod_{i=1}^{n} x_i \right)^\theta 1_{\min x_i > 0} \prod_{i=1}^{n} 1_{\max x_i < 1} \prod_{i=1}^{n} 1_{\theta > 0} \]

:. Log-likelihood \( l(\theta) \equiv \ln \left[ L(\theta|x_1, \ldots, x_n) \right] \)

\[ l(\theta) = n \ln(\theta+1) - \theta \sum_{i=1}^{n} \ln x_i \]

\[ l'(\theta) = \frac{n}{\theta+1} - \sum_{i=1}^{n} \frac{\ln x_i}{\theta+1} \]

:. \( \hat{\theta}_{MLE} = \frac{n}{\sum_{i=1}^{n} \ln x_i} - 1 \)  \hspace{1cm} (1)

Fisher information,

\[ I(\theta) = -E_\theta \left[ \frac{\partial^2}{\partial \theta^2} \ln f(x, \theta) \right] \]

\[ = -E_\theta \left[ \frac{\partial^2}{\partial \theta^2} \ln (f(x, \theta)) \right] \]

\[ = -E_\theta \left[ \frac{\partial^2}{\partial \theta^2} \left[ \ln (\theta+1) - \theta \ln x \right] \right] \]

\[ = \frac{1}{(\theta+1)^2} \]

Now, we know

\[ \sqrt{n} (g(\hat{\theta}_{MLE}) - g(\theta)) \Rightarrow N(0, \frac{[g'(\theta)]^2}{I(\theta)}) \]
Thus, if we choose
\[ g'( \theta) = \sqrt{I(\theta)} \]

Then
\[ \sqrt{n} \left( g(\hat{\theta}_{\text{MLE}}) - g(\theta) \right) \Rightarrow N(0, 1) \]

and we have a pivot.

Thus, in this problem,
\[ g'(\theta) = \frac{1}{\theta + 1} \]
\[ \therefore g(\theta) = \ln(\theta + 1) \]

So,
\[ \sqrt{n} \left( \ln(\hat{\theta} + 1) - \ln(\theta + 1) \right) \Rightarrow N(0, 1) \]

Then,
\[ P_\theta \left[ - \frac{z_{1/2}}{\sqrt{n}} \leq \sqrt{n} \ln \left( \frac{\hat{\theta} + 1}{\theta + 1} \right) \leq \frac{z_{1/2}}{\sqrt{n}} \right] = 1 - \alpha \]

i.e.
\[ P_\theta \left[ \exp \left( -\frac{z_{1/2}}{\sqrt{n}} \right) \leq \frac{\hat{\theta} + 1}{\theta + 1} \leq \exp \left( \frac{z_{1/2}}{\sqrt{n}} \right) \right] = 1 - \alpha \]

i.e.
\[ P_\theta \left[ \frac{\hat{\theta} + 1}{\exp \left( -\frac{z_{1/2}}{\sqrt{n}} \right)} \geq \theta + 1 \geq \frac{\hat{\theta} + 1}{\exp \left( \frac{z_{1/2}}{\sqrt{n}} \right)} \right] = 1 - \alpha \]

Thus, the required C.I is
\[ \left[ \frac{\hat{\theta} + 1}{\exp \left( \frac{z_{1/2}}{\sqrt{n}} \right) - 1}, \frac{\hat{\theta} + 1}{\exp \left( -\frac{z_{1/2}}{\sqrt{n}} \right) - 1} \right] \text{ where } \hat{\theta} \text{ is given by (1) }
(4) \[ x_i = 1 (Y_i = 0) \]

Thus, \( x_i \sim \text{Bernoulli} \left( e^{-\theta} \right) \)

\[ [\text{since } P (Y_i = 0) = \frac{e^{-\theta} \theta^0}{0!} = e^{-\theta}] \]

Now, if \( x_i \sim \text{Bernoulli} \left( \theta \right) \)

Likelihood \( f_n \).

\[ L (\theta \mid x_1, \ldots, x_n) = \prod_{i=1}^{n} p^{x_i} (1-p)^{1-x_i} \]

Log-likelihood,

\[ l (\theta) = \ln p \sum_{i=1}^{n} x_i + \ln (1-p) \sum_{i=1}^{n} (1-x_i) \]

\[ l' (\theta) = \frac{\sum_{i=1}^{n} x_i}{p} - \frac{\sum_{i=1}^{n} (1-x_i)}{1-p} \]

For MLE, \( l' (\hat{\theta}) = 0 \)

\[ \hat{\theta} = \frac{\sum_{i=1}^{n} x_i}{\bar{x}} = \frac{\sum_{i=1}^{n} (1-x_i)}{1-\bar{x}} \]

or, \( \frac{\bar{x}}{\hat{\theta}} = \frac{1-\bar{x}}{1-\hat{\theta}} \)

or, \( \frac{1-\hat{\theta}}{1-\bar{x}} = \frac{1-x}{x} \)

or, \( \frac{1}{\hat{\theta}} - 1 = \frac{1}{x} - 1 \quad \Rightarrow \quad \hat{\theta} = \bar{x} \)
Thus, since \( p = e^{-\theta} \) in this question,
\[
e^{-\hat{\theta}_x} = \frac{1}{x}
\]
(continuous mapping)
\[
\Rightarrow \hat{\theta}_x = -\ln x
\]
where \( \hat{\theta}_x \) is the MLE based on \( X_i \)'s.

Now, if \( Y_i \)'s are available,
\[
L(\Theta / Y_1, \ldots, Y_n) = \prod_{i=1}^{n} \frac{e^{-\Theta} \Theta^{Y_i}}{Y_i^n}
\]
\[
\Rightarrow \ell(\Theta) = -n \Theta + \ln \Theta. \sum_{i=1}^{n} Y_i - \ln (\prod_{i=1}^{n} Y_i !)
\]

For MLE \( \ell'(\hat{\Theta}_Y) = 0 \)
i.e. \[-n + \frac{1}{\hat{\Theta}_Y} \sum_{i=1}^{n} Y_i = 0
\]
\[
\Rightarrow \hat{\Theta}_Y = \bar{Y}, \hat{\Theta}_Y \text{ is the MLE based on } Y_i \text{'s}
\]

Qualitative comparison:

\( Y_i \)'s are the full data, where \( X_i \)'s only tell if \( Y_i \)'s are 0 or not. Thus, \( Y_i \)'s have more information and an estimator based on \( Y_i \)'s will be more efficient, in the
sense that we will require fewer samples to get comparable standard error for our estimate than if we base our estimator on $X_i$'s. This ratio of sample sizes can be computed explicitly, but is not necessary here.
(5) Let $T_i$ be the time between $(i-1)$th and $i$th emissions ($i \geq 1$)

then, $T_i \sim \text{exp (mean = } \beta)$

Let $S_n = \sum_{i=1}^{n} T_i$

Thus, $S_n \sim \text{gamma (n, } \frac{1}{\beta})$

Thus, data available to the physicist = $S_n$.

Now, $\frac{S_n}{\beta} \sim \text{gamma (n, 1)}$

Suppose $c_1$ and $c_2$ denote the $\alpha/2$th and $(1-\alpha/2)$th quantile's of Gamma ($n, 1$)

Thus, $P_n \left[ c_1 \leq \frac{S_n}{\beta} \leq c_2 \right] = 1 - \alpha$

Thus, the required CI is

$[ \frac{S_n}{c_2}, \frac{S_n}{c_1} ]$