Final Exam: Stat 426.

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INFORMATION: Answer 5 out of 6 questions. Each question carries 16 points; your effective score on the final is the minimum of what you get and 70. Half of this score adds up towards your final grade.

(1) (a) Suppose that the radius of a circle is a random variable having the following probability density function.

\[ f(x) = \frac{1}{8}(3x + 1), \quad 0 < x < 2 \]

and 0 otherwise. Determine the probability density function of the area of the circle.

(b) Suppose that \( X \) and \( Y \) are i.i.d. random variables and each has the p.d.f.

\[ f(x) = e^{-x}, \quad x > 0 \]

and

\[ f(x) = 0 \text{ otherwise}. \]

Let

\[ U = X/(X + Y) \text{ and } V = X + Y. \]

Determine the joint density of \( U \) and \( V \). Identify the distribution of \( V \). Are \( U \) and \( V \) independent? (6 + 10 = 16 points)

(2) (i) There are two elevators; one fast and one slow. The expected waiting time for the slow elevator is 3 minutes and the waiting time for the fast elevator is 1 minute. The probability that Sam catches the first elevator is \( 2/3 \) and the chance that she catches the second elevator is \( 1/3 \). What is her expected waiting time?

(ii) Let \( U \) be a random variable following a Poisson\( (n) \) distribution. Show that for large \( n \), the distribution of \( U \) is well-approximated by a normal distribution with mean \( n \) and variance \( n \). (Hint: What is the distribution of the sum of \( n \) independent and identically distributed Poisson random variables?)
(iii) Let $X_1, X_2, \ldots, X_n$ be i.i.d. Gamma($\alpha, \lambda$) random variables, with both $\alpha$ and $\lambda$ unknown. Find a set of sufficient statistics for $(\alpha, \lambda)$.

Find a sufficient statistic for $\alpha$ when $\lambda$ is known to be equal to $\lambda_0$. $(5 + 5 + 6 = 16$ points$)$

(3) (i) Two populations are surveyed with a simple random sample. A sample of size $n_1$ is used for population I with population standard deviation $\sigma_1$ and a sample of size $n_2 = 2n_1$ is used for population II, with population standard deviation $\sigma_2 = 2\sigma_1$. Ignoring finite population corrections, in which sample would you expect the estimate of the population mean to be more accurate? Justify your answer.

(ii) In a city 30% of people are conservatives, 50% are liberals and 20% are independents. In a particular election, 60% of conservatives voted, 80% of liberals voted and 50% of independents voted. If a person is selected at random from the population and it is learnt that they did not vote in the last election, what is the chance that the randomly chosen person is a liberal?

(iii) If $X$ and $Y$ are independent exponential($\lambda$) random variables, show that $X/Y$ has an $F$ distribution and identify the degrees of freedom. (Hint: If $X$ follows exp($\lambda$) what does $2\lambda X$ follow?) $(5 + 5 + 6 = 16$ points$)$

(4) (a) Let $X$ follow a Poi($\theta$) distribution. Define a statistic $T(X)$ as follows:

$$T(X) = 1 \text{ if } X = 0,$$

and

$$T(X) = 0, \text{ otherwise.}$$

(i) Compute the information bound for the variance of $T(X)$ obtained from the information inequality.

(ii) Compute the variance of $T(X)$ and show that the bound obtained from the information inequality is strictly less than the variance. (This shows that the bound is not sharp. Nevertheless, it can be shown that $T(X)$ is the best unbiased estimator of its expectation. The above example illustrates that best unbiased estimators may not always achieve the lower bound given by the information inequality) (Hint: For $x > 0$, $e^x - 1 > x$) $(4 + 5 = 9$ points$)$

(b) Let $X_1, X_2, \ldots, X_n$ be a sample from a population with density

$$f(x, \theta) = \theta (\theta + 1) x^{\theta-1} (1-x) , \ 0 < x < 1, \ \theta > 0.$$
(i) Show that
\[ T_n = \frac{2\overline{X}_n}{1 - \overline{X}_n}, \]
is a MOM estimator of \( \theta \).

(ii) Identify the limit distribution of \( \overline{X}_n \). (4 + 3 = 7 points)

(5) Let \( X_1, X_2, \ldots, X_n \) be i.i.d. Geometric(\( \theta \)). Thus, each \( X_i \) has probability mass function,
\[ f(x, \theta) = (1 - \theta)^{x-1} \theta, \quad \theta > 0, \quad x = 1, 2, 3, \ldots \]

(a) Show that the joint density of \( X = (X_1, X_2, \ldots, X_n) \) which we denote by \( p(x, \theta) \) belongs to the exponential family and that we can take \( T(X) = \sum X_i \). Identify \( c(\theta) \) and \( d(\theta) \).

(b) Compute the information about \( \theta \) (call this \( I_n(\theta) \)) based on the joint density \( p(x, \theta) \). What is the information about \( \theta \) based on 1 observation, say, \( X_1 \)?

(c) Apply the second theorem in the Exponential Families notes (the one on computing MLEs for exponential family models) to derive \( \hat{\theta}_n \), the MLE of \( \theta \).

(d) Identify the limit distribution of \( \sqrt{n}(\hat{\theta}_n - \theta) \) and use this result (or any other results known to you) to show that
\[ \sqrt{n} \left( \frac{1}{\hat{\theta}_n} - \frac{1}{\theta} \right) \to N \left( 0, \frac{1 - \theta}{\theta^2} \right). \]
(3 + 4 + 4 + 5 = 16 points)

(6) Let \( X_1, X_2, \ldots, X_n \) be i.i.d. \( N(\mu_1, \sigma^2) \) and let \( Y_1, Y_2, \ldots, Y_m \) be i.i.d. \( N(\mu_2, \sigma^2) \) and let the \( X_i \)'s and \( Y_j \)'s be independent. Denoting \((X_1, X_2, \ldots, X_n)\) by \( X \) and \((Y_1, Y_2, \ldots, Y_m)\) by \( Y \), we can write the joint likelihood as,
\[
l(X, Y, \mu_1, \mu_2, \sigma^2) = \frac{1}{(2\pi)^{n/2} \sigma^n} \exp \left( -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (X_i - \mu_1)^2 \right) \times \frac{1}{(2\pi)^{m/2} \sigma^m} \exp \left( -\frac{1}{2\sigma^2} \sum_{j=1}^{m} (Y_j - \mu_2)^2 \right).
\]

(i) Show that the MLE's of \((\mu_1, \mu_2, \sigma^2)\) are \((\overline{X}, \overline{Y}, \hat{\sigma}^2)\), where \( \overline{X} \) is the sample average of the \( X_i \)'s, \( \overline{Y} \) is the sample average of the \( Y_j \)'s and
\[ \hat{\sigma}^2 = \frac{\sum_{i=1}^{n} (X_i - \overline{X})^2 + \sum_{j=1}^{m} (Y_j - \overline{Y})^2}{m + n}. \]
**Hint:** Don’t panic. One way to attack this is as follows: Take the log of the likelihood. Fix \( \sigma^2 \) and maximize with respect to \( \mu_1 \) and \( \mu_2 \); use the same techniques as I used in Example 4 of the MLE section of the Methods of Estimation notes. You should be able to show that for fixed \( \sigma^2 \) the MLE’s of \( \mu_1 \) and \( \mu_2 \) are \( \bar{X} \) and \( \bar{Y} \) respectively. You can then plug in these values for \( \mu_1 \) and \( \mu_2 \) into the expression for the log-likelihood and subsequently maximize with respect to \( \sigma^2 \) in the same way done in Example 4, to end up with the desired estimate of \( \sigma^2 \).

(ii) Show that
\[
\bar{X} - \bar{Y} \sim N(\mu_1 - \mu_2, \sigma^2 (1/m + 1/n))
\]
and that
\[
(m + n) \frac{\sigma^2}{\sigma^2} \sim \chi^2_{m+n-2}.
\]

(iii) Use the above results to show that,
\[
D_{m,n} \frac{(\bar{X} - \bar{Y}) - (\mu_1 - \mu_2)}{\sigma} \sim t_{m+n-2},
\]
for some known constant \( D_{m,n} \) (you need to identify this constant); how can you use this result to find a confidence interval for \( (\mu_1 - \mu_2) \)? (5 + 6 + 5 = 16 points)