

On a Dualization of Graphical Gaussian Models:

A Correction Note

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Abstract

We correct two proofs concerning Markov properties for graphs representing marginal independence relations.

Keywords: bi-directed graph, conditional independence, covariance graph, global Markov property, local Markov property, pairwise Markov property.

Running Head: Correction Note: *Dualization of Graphical Models.*

1 Introduction

COX AND WERMUTH (1996, 1993) define covariance graphs which are graphs containing undirected dashed edges (----), representing marginal independence relations.

KAUERMANN (1996) defines the following Markov properties associated with these graphs:

A family \mathcal{P} of distributions over a set V of variables is called

- (i) *pairwise τ - \mathcal{G} Markov*, if $\alpha \perp\!\!\!\perp \beta$ for all α and β not adjacent in \mathcal{G} ,
- (ii) *local τ - \mathcal{G} Markov*, if $\alpha \perp\!\!\!\perp V \setminus (\{\alpha\} \cup \text{bd}(\alpha))$ for all vertices α of \mathcal{G} ,
- (iii) *global τ - \mathcal{G} Markov*, if $B \perp\!\!\!\perp C \mid A$ whenever B and C are separated by $D = V \setminus (A \cup B \cup C)$ in \mathcal{G} , where A , B and C are mutually disjoint, and A , B are non-empty.

For definitions of graphical terms see LAURITZEN (1996), KAUERMANN (1996).

These properties may be viewed as a ‘dualization’ of the classical Markov properties associated with undirected graphs:

A family \mathcal{P} of distributions over a set V of variables is called

- (i*) *pairwise θ - \mathcal{G} Markov*, if $\alpha \perp\!\!\!\perp \beta \mid (V \setminus \{\alpha, \beta\})$ for all α and β not adjacent in \mathcal{G} ,
- (ii*) *local θ - \mathcal{G} Markov*, if $\alpha \perp\!\!\!\perp V \setminus (\{\alpha\} \cup \text{bd}(\alpha)) \mid \text{bd}(\alpha)$ for all vertices α of \mathcal{G} ,
- (iii*) *global θ - \mathcal{G} Markov*, if $B \perp\!\!\!\perp C \mid A$ whenever B and C are separated by A in \mathcal{G} , where A , B and C are disjoint, and A , B are non-empty.

Kauermann then restricts attention to those distributions which satisfy the following property for all disjoint subsets A, B, C :

$$A \perp\!\!\!\perp B \text{ and } A \perp\!\!\!\perp C \quad \Rightarrow \quad A \perp\!\!\!\perp (B \cup C). \quad (1)$$

While this is a strong assumption, note that this is readily satisfied if the underlying distribution is Gaussian. In this note we correct errors present in two proofs of results stated by Kauermann, which relate these Markov properties.

2 Equivalence of τ - \mathcal{G} Markov properties

Proposition 2.2 of KAUERMANN (1996) states:

Proposition 2.2 *Let \mathcal{P} be a family of distributions that fulfill (1) for all disjoint subsets A, B, C . The following statements are then equivalent: (i) \mathcal{P} fulfills the pairwise τ - \mathcal{G} Markov property; (ii) \mathcal{P} fulfills the local τ - \mathcal{G} Markov property; (iii) \mathcal{P} fulfills the global τ - \mathcal{G} Markov property.*

The implication (iii) \Rightarrow (ii) follows by definition while (ii) \Rightarrow (i) follows from the implication $A \perp\!\!\!\perp (B \cup C) \Rightarrow A \perp\!\!\!\perp B$, as Kauermann notes. Thus the only implication remaining to be proved is (i) \Rightarrow (iii).

Kauermann's proof considers B and C separated by $D = V \setminus (A \cup B \cup C)$. He then defines $A_1 \subseteq A$ to be the set of all vertices in A which are not adjacent to B , and $A_2 \equiv A \setminus A_1$. Subsequently Kauermann argues that there is no vertex $\alpha_1 \in A_1$ which is adjacent to a vertex in A_2 . However, this is false, as shown by the following graph:

$$\alpha_1 \text{ ---- } \alpha_2 \text{ ---- } \beta \text{ ---- } \delta \text{ ---- } \gamma$$

If $A = \{\alpha_1, \alpha_2\}$, $B = \{\beta\}$ and $C = \{\gamma\}$, then C and B are separated by $D = V \setminus (A, B, C) = \{\delta\}$. However, $A_1 = \{\alpha_1\}$, $A_2 = A \setminus A_1 = \{\alpha_2\}$, but α_1 and α_2 are adjacent.

We provide here a corrected proof:

Proof: Suppose that B and C are separated by $D = V \setminus (A \cup B \cup C)$. Let $A_C^* = \{\alpha \mid \alpha \in A \text{ and there is a path from } \alpha \text{ to a vertex } \gamma \in C \text{ on which every vertex is in } A\}$. Note that this set contains those vertices in A that are adjacent to C . Let $A^* = A \setminus A_C^*$. It follows that no vertex $\beta \in B$ is adjacent to $\alpha \in A_C^*$, because in this case B and C would not be separated by D . Thus $A_C^* \perp\!\!\!\perp B$, by (i) and (1). Likewise, there is no vertex $\alpha \in A_C^*$ adjacent to a vertex $\alpha' \in A^*$ because in this case there would be a path from α' to a vertex $\gamma \in C$, and hence $\alpha' \in A_C^*$, which is a contradiction. Thus $A_C^* \perp\!\!\!\perp A^*$, once again by (i) and (1). It now follows by (1) that $A_C^* \perp\!\!\!\perp B \cup A^*$. Similarly, no vertex $\alpha \in A^*$ is adjacent to a vertex in $\gamma \in C$, and hence $C \perp\!\!\!\perp A^*$. Since C and B are separated by D , no vertex $\beta \in B$ is adjacent to $\gamma \in C$, hence $C \perp\!\!\!\perp B$. Thus $C \perp\!\!\!\perp (B \cup A^*)$, and so $(C \cup A_C^*) \perp\!\!\!\perp (B \cup A^*)$, by two applications of (1). For random vectors X, Y, Z and W ,

$$X \perp\!\!\!\perp (Y, Z) \mid W \Rightarrow X \perp\!\!\!\perp Y \mid (Z, W),$$

where W may be empty (DAWID 1979). Two applications of this result enable us to deduce that $C \perp\!\!\!\perp B \mid A$ from the fact that $(C \cup A_C^*) \perp\!\!\!\perp (B \cup A^*)$, as required. \square

3 Relating τ - \mathcal{G} and θ - \mathcal{G} global Markov properties

Let $\mathcal{N}_\theta(\mathcal{G})$, $\mathcal{N}_\tau(\mathcal{G})$ denote the sets of Normal distributions obeying, respectively, the θ and τ Markov properties. For normal distributions, local, global and pairwise properties are equivalent. FRYDENBERG (1990) shows that if $B \perp\!\!\!\perp C \mid A$ for every distribution in $\mathcal{N}_\theta(\mathcal{G})$ then B and C are separated by A in \mathcal{G} . This property is referred to as θ - Markov perfectness of the family

$\mathcal{N}_\theta(\mathcal{G})$. In Theorem 3.1 Kauermann proves that $\mathcal{N}_\tau(\mathcal{G})$ is τ -Markov perfect; in other words, if $B \perp\!\!\!\perp C \mid A$ for every distribution P in $\mathcal{N}_\theta(\mathcal{G})$, then B and C are separated by $D = V \setminus (A \cup B \cup C)$ in \mathcal{G} .

In Theorem 3.2 Kauermann considers for which graphs, $\mathcal{N}_\tau(\mathcal{G}) = \mathcal{N}_\theta(\mathcal{G})$:

Theorem 3.2 $\mathcal{N}_\tau(\mathcal{G}) = \mathcal{N}_\theta(\mathcal{G})$ if and only if \mathcal{G} is either complete or consists of unconnected complete subgraphs.

The ‘if’ part follows from the fact that the inverse of a block-diagonal matrix is block diagonal with the same blocks, as noted by Kauermann.

In proving the other direction Kauermann argues that if B and C are separated by A and by $D = V \setminus (A \cup B \cup C)$ then B and C are unconnected subgraphs. This is false as shown by the following graph:

$$\beta \text{ ---- } \alpha \text{ ---- } \delta \text{ ---- } \gamma$$

$B = \{\beta\}$ and $C = \{\gamma\}$ are separated by $A = \{\alpha\}$ and by $D = \{\delta\}$.

We provide here a correct proof.

Proof: Suppose that $\mathcal{N}_\tau(\mathcal{G}) = \mathcal{N}_\theta(\mathcal{G})$, but \mathcal{G} is not a union of disconnected cliques. The claim is trivial if $|V| \leq 2$, so suppose $|V| > 2$. If, for every triple of vertices such that $\alpha \text{ ---- } \beta \text{ ---- } \gamma$, we also have, $\alpha \text{ ---- } \gamma$, then by induction, any two vertices that are connected by a path are also adjacent, hence in this case \mathcal{G} is a union of disconnected cliques. Consequently there is at least one triple $\alpha \text{ ---- } \beta \text{ ---- } \gamma$ with α and γ not adjacent. By the pairwise θ - \mathcal{G} Markov property we have that $\alpha \perp\!\!\!\perp \gamma \mid (V \setminus \{\alpha, \gamma\})$ for every distribution in $\mathcal{N}_\theta(\mathcal{G}) = \mathcal{N}_\tau(\mathcal{G})$. But now by KAUERMANN (1996) Theorem 3.1, referred to above, it follows that α and γ are separated by $V \setminus (\{\alpha, \gamma\} \cup (V \setminus \{\alpha, \gamma\})) = \emptyset$

in \mathcal{G} . But this is a contradiction since there is a path from α to γ via β in \mathcal{G} . Hence $\mathcal{N}_\tau(\mathcal{G}) \neq \mathcal{N}_\theta(\mathcal{G})$. \square

4 Afterword

Note that the independence models represented here by undirected graphs with dashed edges may also be represented via graphs containing bi-directed edges (\leftrightarrow) via a natural extension of Pearl's d-separation criterion; see PEARL (1988), SPIRITES ET AL. (1998), KOSTER (1999), RICHARDSON (2002), RICHARDSON (2003).

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