Midterm 1 – Statistics 426.

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Announcement: This exam is open book, open notes and carries 24 points, but the maximum that you can score is 20. Problem 2(iv) is more difficult than the others and I would advocate trying it last.

1 Let $X_1$ and $X_2$ be i.i.d. standard normal random variables (a standard normal random variable is normal with mean 0 and variance 1).

(i) Write down the joint density $f(x_1, x_2)$ of $(X_1, X_2)$.

(ii) Define

$$W_1 = \frac{\sqrt{3}X_1 + X_2}{2} \quad \text{and} \quad W_2 = \frac{X_1 - \sqrt{3}X_2}{2}.$$ 

Compute the joint density of $(W_1, W_2)$ and the marginal densities of $W_1$ and $W_2$? What is the probability that $W_1^2 + W_2^2 < 1$? (2 + (5+2+2) = 11 points).

2 Consider an urn that contains three balls marked 1, 2 and 3. I pick a ball out of the urn at random. Let $X_1$ denote the number on this ball. Without returning this ball back to the urn, I draw another ball from the urn, again at random. Let $X_2$ denote the number on this ball. There is now one ball remaining in the urn and I pick it out. Let $X_3$ denote the number on this ball.

(i) Clearly, I always know the value of $X_3$ even before picking the third ball out of the urn, if I have noted the numbers $X_1$ and $X_2$. Why is then $X_3$ still a random variable?

(ii) Compute $P(X_1 = 1, X_2 = 2, X_3 = 3)$. What is the value of $P(X_1 = i, X_2 = j, X_3 = k)$, where $(i, j, k)$ is some permutation of $(1, 2, 3)$?

(iii) Compute $E(X_1 + X_2 + X_3)$ and $\text{Var}(X_1 + X_2 + X_3)$.

(iv) It is not difficult to show that $X_1$, $X_2$ and $X_3$ have the same marginal distributions and that the joint distributions of $(X_1, X_2)$, $(X_2, X_3)$ and $(X_1, X_3)$ are the same. Using
this result and the value of $\text{Var}(X_1 + X_2 + X_3)$ obtained in (iii) (or otherwise) show that $\text{Cov}(X_1, X_2) < 0$. ($3 + 3 + 3 + 4 = 13$)

**Helpful results.**

(a) Recall that

$$P(A_1 \cap A_2 \cap A_3) = P(A_1) P(A_2 | A_1) P(A_3 | A_1 \cap A_2).$$

(b) Also recall that,

$$\text{Var} \left( \sum_{i=1}^{n} X_i \right) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{i<j} \text{Cov}(X_i, X_j).$$