Midterm 1 Solution Sketch: Stat 426.

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(1) (i) The joint density of \( (X_1, X_2) \) is,

\[
f(x_1, x_2) = \frac{1}{2\pi} \exp \left( -\frac{x_1^2 + x_2^2}{2} \right).
\]

(ii) To compute the joint density of \( (W_1, W_2) \) we need to express \( (X_1, X_2) \) as a function of \( (W_1, W_2) \). We have,

\[2 W_1 = \sqrt{3} X_1 + X_2 \text{ and } 2 W_2 = X_1 - \sqrt{3} X_2.\]

Therefore,

\[2 \sqrt{3} W_1 + 2 W_2 = 3 X_1 + \sqrt{3} X_2 + X_1 - \sqrt{3} X_2 = 4 X_1,
\]

showing that

\[X_1 = \frac{\sqrt{3} W_1 + W_2}{2}.
\]

Similar manipulations, or direct substitution now gives,

\[X_2 = \frac{W_1 - \sqrt{3} W_2}{2}.
\]

The joint density of \( (W_1, W_2) \) is

\[
f_W(w_1, w_2) = \frac{1}{2\pi} \exp \left( -\frac{(\sqrt{3} w_1 + w_2)^2/4 + (w_1 - \sqrt{3} w_2)^2/4}{2} \right) J,
\]

where

\[J = \left| \frac{\partial X_1}{\partial W_1} \frac{\partial X_2}{\partial W_2} - \frac{\partial X_1}{\partial W_2} \frac{\partial X_2}{\partial W_1} \right|.
\]

Check that \( J = 1 \) and that

\[(\sqrt{3} w_1 + w_2)^2/4 + (w_1 - \sqrt{3} w_2)^2/4 = w_1^2 + w_2^2.
\]
It follows that
\[ f_W(w_1, w_2) = \frac{1}{2\pi} \exp \left( -\frac{w_1^2 + w_2^2}{2} \right) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{w_1^2}{2} \right) \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{w_2^2}{2} \right). \]

It follows that \( W_1 \) and \( W_2 \) are i.i.d. \( N(0, 1) \) random variables.

The chance that \( W_1^2 + W_2^2 < 1 \) is simply the chance that a \( \chi^2 \) random variable is less than 1. But a \( \chi^2 \) is just an exponential \( 1/2 \) random variable. Hence this probability can be directly evaluated. (How?)

(2) (i) The value of \( X_3 \) is known only conditionally on the values of \( X_1 \) and \( X_2 \), but not otherwise. Note that \( X_3 = 6 - (X_1 + X_2) \); since \( X_1 + X_2 \) is a non-degenerate random variable, so clearly is \( X_3 \) and \( P(X_3 = i) = P(X_1 + X_2 = 6 - i) \).

(ii) \[
P(X_1 = i, X_2 = j, X_3 = k) = P(X_1 = i) P(X_2 = j|X_1 = i) P(X_3 = k|X_1 = i, X_2 = j)
= \frac{1}{3} \times \frac{1}{2} \times 1
= \frac{1}{6}.
\]

(iii) Note that \( X_1 + X_2 + X_3 = 6 \) identically, i.e. with probability 1. Thus \( E(X_1 + X_2 + X_3) = 6 \) and \( \text{Var}(X_1 + X_2 + X_3) = 0 \).

(iv) \[
\text{Var}(X_1 + X_2 + X_3) = \sum_{i=1}^{n} \text{Var}(X_i) + 2 \sum_{i<j} \text{Cov}(X_i, X_j).
\]

Since the \( X_i \)'s have the same marginal distributions and the pairs \((X_1, X_2), (X_1, X_3)\) and \((X_2, X_3)\) are identically distributed we have,

\[
\text{Var}(X_1 + X_2 + X_3) = 3 \text{Var}(X_1) + 6 \text{Cov}(X_1, X_2).
\]

But \( \text{Var}(X_1 + X_2 + X_3) = 0 \) showing that,

\[
3 \text{Var}(X_1) + 6 \text{Cov}(X_1, X_2) = 0
\]

which shows that,

\[
-\text{Var}(X_1) = 2 \text{Cov}(X_1, X_2).
\]

Hence \( \text{Cov}(X_1, X_2) < 0 \).