1. Given that $X$ is uniform on $(0,1)$ and $Y$ is $\exp(1)$, we need to find the distribution of $U = X + Y$ and $V = X/Y$.

We will use the Jacobian theorem. The joint density of $(X,Y)$ is

$$f(x,y) = e^{-y}, \text{ for } 0 < x < 1, y > 0,$$

and is 0 otherwise. Check that

$$X = \frac{UV}{1+V} \text{ and } Y = \frac{U}{1+V}.$$ 

Also, check that the Jacobian of $(U,V)$ with respect to $(X,Y)$ is

$$\left| \begin{vmatrix} \frac{\partial X}{\partial U} & \frac{\partial X}{\partial V} \\ \frac{\partial Y}{\partial U} & \frac{\partial Y}{\partial V} \end{vmatrix} \right| = \frac{U}{(1+V)^2}.$$ 

Also, note that the transformed variables $U$ and $V$ can both vary between 0 and $\infty$ but since $X < 1$ we have $U < (1 + V)/V$. Using the Jacobian theorem, the joint density of $(U,V)$ is given by:

$$f_{U,V}(u,v) = e^{-u/(1+v)} \frac{1}{(1+v)^2}, \text{ } 0 < u,v < \infty, \text{ } u < (1+v)/v,$$

and is 0 otherwise. One can now find the marginals by integrating. I will find the marginal of $V$. Clearly, the marginal of $V$ is,

$$f_V(v) = \int_{0}^{(1+v)/v} \frac{u}{(1+v)^2} e^{-u/(1+v)} \, du.$$ 

Making a change of variable,

$$t = \frac{u}{1+v}$$

and noting that as $u$ runs from 0 to $(1+v)/v$, $t$ runs from 0 to $1/v$ and that $dt = du/(1+v)$ we easily get,

$$f_V(v) = \int_{0}^{1/v} t e^{-t} \, dt,$$

which on integration by parts gives,

$$f_V(v) = 1 - e^{-1/v} \left( 1 + \frac{1}{v} \right).$$

2. In class, I derived the following. If $f_{X,Y}(x,y)$ is the joint density of $(X,Y)$ (where $X$ and $Y$ are non-negative random variables) and if $U = XY$ and $V = X/Y$, then the joint density of $(U,V)$ is,

$$f_{U,V}(u,v) = f_{X,Y}(\sqrt{uv}, \sqrt{u/v}) \frac{1}{2v}.$$
This is a direct application of the Jacobian theorem.

Identify $X_1$ and $X_2$ with $X$ and $Y$ respectively. Note that $Z$ in Problem 2 is just $V$. The joint density of $(X,Y)$ in Problem 2 is,

$$f_{X,Y}(x, y) = \lambda_1 \lambda_2 e^{-\lambda_1 x - \lambda_2 y}.$$  

The joint density of $(U,V)$ is

$$f_{U,V}(u, v) = \lambda_1 \lambda_2 e^{-\lambda_1 \sqrt{uv} - \lambda_2 \sqrt{u/v}} \frac{1}{2v}.$$  

To compute the marginal density of $V$, we just need to integrate out $U$. We have,

$$f_V(v) = \int_0^\infty \lambda_1 \lambda_2 \exp \left(-\sqrt{u} \frac{\lambda_1 v + \lambda_2}{\sqrt{v}}\right) \frac{1}{2v} \, du.$$  

Let,

$$K(v) = \frac{\lambda_1 v + \lambda_2}{\sqrt{v}}.$$  

Set $u = w^2$. Then $du = 2w \, dw$ and the above integral reduces to

$$f_V(v) = \frac{\lambda_1 \lambda_2}{2v} \int_0^\infty \exp(-K(v) \, w) \, 2w \, dw$$  

$$= \frac{\lambda_1 \lambda_2}{v K(v)} \int_0^\infty K(v) \exp(-K(v) \, w) \, w \, dw$$  

$$= \frac{\lambda_1 \lambda_2}{v K(v)^2},$$  

where this last step follows on noting that

$$\int_0^\infty K(v) \exp(-K(v) \, w) \, w \, dw$$

is just the mean of an exponential random variable with parameter $K(v)$ and is therefore equal to $1/K(v)$. On simplifying, we get,

$$f_V(v) = \frac{\lambda_1 \lambda_2}{v K(v)^2} = \frac{\lambda_1 \lambda_2}{(\lambda_1 v + \lambda_2)^2}.$$  

This is the marginal density of $V$. The probability that $V$ is less than 1 which is the prob. that $X_1 < X_2$ is obtained by integrating the above density over the region $(0,1)$. It is in general NOT equal to $1/2$, unless $\lambda_1 = \lambda_2$.

3. The expression for the joint density follows again as an application of what I derived in class. The only thing that you need to be careful about is the domain of $U = XY$ and $V = X/Y$. Note that $X \geq 1$ and $Y \geq 1$. This implies that $U \geq 1$ and $0 < V < \infty$. Note also that
$UV \geq 1$ and $U/V \geq 1$, which is equivalent to saying that $U \geq \max(V, 1/V)$. In other words, the density of $(U, V)$ is concentrated on $S \equiv \{(u, v) : 0 \leq v, 1 \leq u, u \geq \max(v, 1/v)\}$. The condition $u \geq \max(v, 1/v)$ just means that $u \geq v$ when $v \geq 1$ and $u \geq \frac{1}{v}$ when $v < 1$. Check that the joint density of $(U, V)$ is simply,

$$f_{U,V}(u, v) = \frac{1}{u^2} \frac{1}{2v}, \ (u, v) \in S,$$

and is 0 otherwise. You can now compute the marginals of $U$ and $V$. To find the marginal of $V$, consider two cases, (i) $v < 1$ and (ii) $v \geq 1$. If (i) holds,

$$f_V(v) = \int_{1/v}^{\infty} \frac{1}{u^2} \frac{1}{2v} = \frac{1}{2},$$

by straightforward computation. In case (ii),

$$f_V(v) = \int_{v}^{\infty} \frac{1}{u^2} \frac{1}{2v} = \frac{1}{2v^2},$$

once again by straightforward computation.

4. if $U = X + Y$ and $V = X - Y$, then $X = (U + V)/2$ and $Y = (U - V)/2$ and the Jacobian of $(X, Y)$ with respect to $(U, V)$ is $1/2$. Check that the density of $(U, V)$ is identically $1/2$ over the region where $0 < U + V < 2, 0 < U - V < 2, 0 < U < 2$ and $-1 < V < 1$. This is precisely the square formed by the points $(0, 0), (1, 1), (2, 0), (1, -1), (0, 0)$.

5. Note that $X_1 = \log Y_2$ and $X_2 = Y_1 - \log Y_2$. The Jacobian of $(X_1, X_2)$ with respect to $(Y_1, Y_2)$ is just $\frac{1}{Y_2}$. The domain of $(Y_1, Y_2)$ is $S \equiv \{(y_1, y_2) : 0 \leq y_1, 1 \leq y_2, \log y_2 \leq y_1\}$. The joint density of $(X_1, X_2)$ is

$$f_{X_1,X_2}(x_1, x_2) = \lambda^2 \exp(-\lambda(x_1 + x_2)).$$

The joint density of $(Y_1, Y_2)$ is

$$f_{Y_1,Y_2}(y_1, y_2) = \lambda^2 \exp(-\lambda y_1) \frac{1}{y_2}, (y_1, y_2) \in S$$

and 0 otherwise.