On the ergodicity and mixing of max–stable processes

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Outline

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- Representations
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Max–stable processes

• $\{X_t\}_t$ is max–stable if all f.d.d. are max–stable. The marginals of X are thus either *Fréchet*, *Gumbel* or *negative Fréchet*.

• Recall that X is $\alpha-$Fréchet ($\alpha > 0$) with scale $\sigma > 0$, if

$$\mathbb{P}\{X \leq x\} = \exp\{-(x/\sigma)^{-\alpha}\}, \quad x \in (0, \infty).$$

Def $\{X_t\}$ is an $\alpha-$*Fréchet process* if for all $a_i > 0$ and t_i, $i = 1, \ldots, d$, the *max–linear combination*

$$\max\{a_i X_{t_i},\ 1 \leq i \leq d\} = \bigvee_{i=1}^{d} a_i X_{t_i},$$

is $\alpha-$Fréchet.

Thm [de Haan (1978)] A process X with $\alpha-$Fréchet marginals is max–stable *if and only if* it is an $\alpha-$Fréchet process.

• The class of Fréchet processes is sufficient for the study of ergodicity.

Why ergodicity?

Let $\{X_k,\ k = 1, 2, \ldots, n\}$ is a sample from a stationary max–stable process.

• Ergodicity implies that

$$\frac{1}{n}\sum_{k=1}^{n} h(X_k) \xrightarrow{a.s.} \mathbb{E}h(X_1), \quad \text{as } n \to \infty,$$

for $\mathbb{E}|h(X_1)|$ finite.

• ... or, more generally, that

$$\frac{1}{n-m}\sum_{k=0}^{n-m-1} h(X_{k+1}, \cdots, X_{k+m}) \xrightarrow{a.s.} \mathbb{E}h(X_1, \cdots, X_m), \quad \text{as } n \to \infty,$$

for $\mathbb{E}|h(X_1, \cdots, X_m)|$ finite.

• Thus ergodicity has important statistical implications.

• To characterize the ergodic properties of max-stable processes one needs suitable representations.

Representations

Thm [the de Haan spectral representation] *If* $X = \{X_t\}_{t\in\mathbb{R}}$ *is continuous in probability* $\alpha-$*Fréchet process, then*

$$\{X_t\} \stackrel{fdd}{=} \{ \bigvee_{k=1}^{\infty} f_t(U_k)/\epsilon_k^{1/\alpha} \},$$

for a Poisson point process $\{(U_k, \epsilon_k)\}$ *in* $(0,1)\times(0,\infty)$ *with intensity* $\rho(du)\times dx$ *and deterministic functions* $f_t(\cdot) \geq 0$ *with* $\int_0^1 f_t(u)^\alpha \rho(du) < \infty$*.*

• Equivalently, one can use *extremal integrals*:

$$\{X_t\}_t \stackrel{fdd}{=} \{ {}^{e}\!\!\int_{(0,1)} f_t(u) M_\alpha(du) \}_t$$

where M_α is an $\alpha-$Fréchet sup–measure on $(0,1)$ with control measure $\rho(du)$.

Intuition: The random measure M_α:

1. assigns independent $\alpha-$Fréchet variables to disjoint intervals.

2. is max–additive instead of additive: $M_\alpha(A\cup B) = M_\alpha(A) \vee M_\alpha(B)$.

Representations (cont'd)

Def M_α *is* $\alpha-$*Fréchet sup–measure on* $(E,\mathcal{E})$ *with control measure* μ*, if: (i) independently scattered (ii)* $\mathbb{P}\{M_\alpha(A)\le x\} = \exp\{-\mu(A)x^{-\alpha}\},\ x>0$ *(iii)* $\sigma-$*sup–additive:*

$$M_\alpha(\cup_{n=1}^\infty A_n) = \bigvee_{n=1}^\infty M_\alpha(A_n), \quad \textit{almost surely.}$$

• Analogous to the $\alpha-$stable measures in the sum–stable case (see Samorodnitsky and Taqqu (1994)).

Intuition: Let $A\cap B=\emptyset$ and $a,b>0$. By independence, for $x>0$

$$\begin{aligned}\mathbb{P}\{aM_\alpha(A)\vee bM_\alpha(B)\le x\} &= \exp\{-\mu(A)(x/a)^{-\alpha}\}\exp\{-\mu(B)(x/b)^{-\alpha}\}\\ &= \exp\{-(a^\alpha\mu(A)+b^\alpha\mu(B))x^{-\alpha}\}.\end{aligned}$$

Representations (cont'd)

Extremal integral: For $f(u) = \sum_{k=1}^{n} a_k 1_{A_k}(u),\ a_k \geq 0$,

$$I(f) := {}^{e}\!\!\int_E f(u) M_\alpha(du) := \bigvee_{k=1}^{n} a_k M_\alpha(A_k).$$

Properties:

1. For simple $f(u) \geq 0$,

$$\mathbb{P}\{I(f) \leq x\} = \exp\{-\int_E f^\alpha d\mu \cdot x^{-\alpha}\},\ x > 0.$$

2. For constants $a, b > 0$, and simple $f(u), g(u) \leq 0$:

$$I(af \vee bg) = aI(f) \vee bI(g).$$

3. For simple $f_i(u) \geq 0,\ 1 \leq i \leq d,\ I(f_i),\ 1 \leq i \leq d$ are independent iff $f_i, 1 \leq i \leq d$ have disjoint supports ($\mu-$a.e.).

- ${}^{e}\!\!\int_E f dM_\alpha$ can be defined for all measurable $f(u) \geq 0$, with $\int_E f^\alpha d\mu < \infty$.

Fréchet spaces

Notation: For $\alpha-$Fréchet ξ, $\mathbb{P}\{\xi \le x\} = \exp\{-cx^{-\alpha}\}$, set $\|\xi\|_\alpha := c^{1/\alpha}$, so that

$$\| {}^{e}\!\!\int_E f dM_\alpha \|_\alpha = \Big(\int_E f^\alpha d\mu\Big)^{1/\alpha}.$$

• The collection of all extremal integrals $I(f) = {}^{e}\!\!\int_E f dM_\alpha,\ f \in L_+^\alpha(\mu)$ is closed w.r.t. *max–linear combinations*:

$$a_1 I(f_1) \vee \cdots \vee a_n I(f_n) = I(a_1 f_1 \vee \cdots \vee a_n f_n), \quad \forall a_i \ge 0.$$

∘ They are *jointly Fréchet* or a *Fréchet space* (like Gaussian spaces).

A natural metric: For *Fréchet spaces*:

$$\rho_\alpha(\xi, \eta) = 2\|\xi \vee \eta\|_\alpha^\alpha - \|\xi\|_\alpha^\alpha - \|\eta\|_\alpha^\alpha,$$

is a metric, which metrizes the convergence in probability.

• For $\xi = {}^{e}\!\!\int_E f dM_\alpha$ and $\eta = {}^{e}\!\!\int_E g dM_\alpha,\ f, g \in L_+^\alpha(\mu)$

$$\rho_\alpha(\xi, \eta) = 2\int_E f^\alpha \vee g^\alpha d\mu - \int_E f^\alpha d\mu - \int_E g^\alpha d\mu = \int_E |f^\alpha - g^\alpha| d\mu.$$

Stationary max–stable processes: pistons and max–linear isometries

Now for any $f_t \in L^\alpha_+(\mu)$

$$X_t = \int^e_E f_t(u) M_\alpha(du), \quad -\infty < t < \infty$$

is a max–stable process.

Thm [de Haan and Pickands (1986)] If $X = \{X_t\}$ is stationary and continuous in probability, then one can take $E = [0, 1]$ and

$$f_t = \Gamma_t(f_0),$$

where $\{\Gamma_t\}$ is a group of pistons.

- The pistons are precisely the max–linear isometries of $L^\alpha_+([0, 1])$.

Def *For measure spaces* $(E, \mathcal{E}, \mu)$ *and* $(F, \mathcal{F}, \nu)$, *the map* $U : L^\alpha_+(\mu) \to L^\alpha_+(\nu)$ *is a max–linear isometry (m.l.i.) if: (i)* $U(af \vee bg) = aU(f) \vee bU(g)$, *for all* $a, b \geq 0$ *and* $f, g \in L^\alpha_+(\mu)$ *(ii)* $\int_E f^\alpha d\mu = \int_F U(f)^\alpha d\nu$, *for all* $f \in L^\alpha_+(\mu)$.

Stationary max–stable processes (cont'd)

• Max–linear isometries $U : L^{\alpha}_{+}(\mu) \to L^{\alpha}_{+}(\mu)$ preserve the metric ρ_α:

$$\rho_\alpha(I(f), I(g)) = \rho_\alpha(I(Uf), I(Ug)).$$

• For a group of m.l.i. $\{U_t\}_{t\in\mathbb{R}}$ on $L^{\alpha}_{+}(\mu)$ and $f_0 \in L^{\alpha}_{+}(\mu)$, the process

$$X_t := {}^{e}\!\!\int_E U_t(f_0) dM_\alpha, \quad t \in \mathbb{R}, \qquad (1)$$

is a (strictly) stationary $\alpha-$Fréchet process.

• We focus on stationary $\alpha-$Fréchet processes as in (1), including, for example (but not limited to):

∘ the moving maxima:

$$X_t := \int_{\mathbb{R}} f_0(t+u) M_\alpha(du), \quad t \in \mathbb{R}.$$

∘ ... or, more generally, the mixed moving maxima:

$$X_t := \int_{\mathbb{R}\times V} f_0(t+u, v) M_\alpha(du, dv), \quad t \in \mathbb{R}.$$

Problem formulation

The process: Let $X = \{X_t\}_{t\in\mathbb{R}}$:

$$X_t := \int_E^e U_t(f)dM_\alpha, \quad -\infty < t < \infty,$$

where $f \in L_+^\alpha(\mu)$ and $\{U_t\}$ is a group of m.l.i. on $L_+^\alpha(\mu)$.

Ergodicity: we look for *necessary and sufficient conditions* (on $\{U_t\}$ and f) for

$$\xi_T(h) := \frac{1}{T}\int_0^T h \circ S_\tau(X)d\tau \xrightarrow{a.s.} \text{const}, \quad \text{as } T \to \infty,$$

for all bounded Borel measurable $h : \mathbb{R}^\mathbb{R} \to \mathbb{R}^\mathbb{R}$, where $S_\tau(\{x_t\}) = \{x_{t+\tau}\}$.

Mixing: we look for *necessary and sufficient conditions* (on $\{U_t\}$ and f) for

$$\mathbb{P}(A \cap B_\tau) \longrightarrow \mathbb{P}(A)\mathbb{P}(B), \quad \text{as } \tau \to \infty,$$

for all $A \in \sigma\{X_t, -\infty < t \le 0\}$ and $B \in \sigma\{X_t,\ 0 < t < \infty\}$, with $B_\tau = S_\tau(B)$.

Technicality: $\{X_t\}$ has a measurable modification if and only if it is continuous in probability (S. (2007)).

General Results: ergodicity

Thm [S. (2007)] $X = \{X_t\}_{t\in\mathbb{R}}$ *is ergodic, if and only if, for some (any)* $p > 0$,

$$\frac{1}{T}\int_0^T \|U_\tau g \wedge g\|^p_{L^\alpha(\mu)} d\tau \longrightarrow 0, \quad \text{as } T \to \infty,$$

for all $g \in F := \overline{\vee - \mathit{span}}\{U_t f,\ t \in \mathbb{R}\}$.

• Here $F \subset L^\alpha_+(\mu)$ contains all positive max–linear combinations of $U_t f$'s and is closed in the metric ρ_α.

◦ Namely, all limits in ρ_α of the max–linear combinations

$$a_1 U_{t_1}(f) \vee \cdots \vee a_n U_{t_n}(f), \quad \forall a_i \ge 0,\ t_i \in \mathbb{R}.$$

General Results: mixing

Thm [S. (2007)] $X = \{X_t\}_{t\in\mathbb{R}}$ *is mixing, if and only if,*

$$\|U_\tau h \wedge g\|_{L_+^\alpha(\mu)} \longrightarrow 0, \quad \text{as } \tau \to \infty,$$

for all $g \in F_-$ *and* $h \in F_+$, *where* $F_- = \overline{\vee - \text{span}}\{U_t f,\ -\infty < t \le 0\}$ *and* $F_- = \overline{\vee - \text{span}}\{U_t f,\ 0 < t < \infty\}$.

• The proofs borrow ideas from the $\alpha-$(sum)stable case – Cambanis, Hardin and Weron (1987).

Idea: $\|f \wedge g\|_{L_+^\alpha(\mu)}$ 'measures' the *dependence* b/w $\xi = {}^{e}\!\!\int_E f dM_\alpha$ and $\eta = {}^{e}\!\!\int_E g dM_\alpha$.

A 'natural' measure of dependence

Def *For jointly $\alpha-$Fréchet ξ and η,*

$$d_\alpha(\xi,\eta) := \|\xi\|_\alpha^\alpha + \|\eta\|_\alpha^\alpha - \|\xi \vee \eta\|_\alpha^\alpha,$$

is a measure of the dependence between ξ and η.

- ξ and η are independent if and only if $d_\alpha(\xi,\eta) = 0$.
- If $\xi = {}^e\!\!\int_E f dM_\alpha$ and $\eta = {}^e\!\!\int_E g dM_\alpha$ then

$$d_\alpha(\xi,\eta) = \int_E f^\alpha d\mu + \int_E g^\alpha d\mu - \int_E f^\alpha \vee g^\alpha d\mu = \int_E f^\alpha \wedge g^\alpha d\mu$$

Thm [S. (2007)] *A continuous in probability stationary $\alpha-$Fréchet process $X = \{X_t\}_{t\in\mathbb{R}}$ is mixing if and only if $d_\alpha(X_t, X_0) \to 0$, as $t \to \infty$.*

- The condition $d_\alpha(X_\tau, X_0) \to 0$, $\tau \to 0$ is easy to check.
- One and the same process $X = \{X_t\}$ may have different representations $(f, \{U_t\}, M_\alpha)$, but has only one dependence function

$$d_\alpha(\tau) = d_\alpha(X_\tau, X_0) = \|X_\tau\|_\alpha^\alpha + \|X_0\|_\alpha^\alpha - \|X_\tau \vee X_0\|_\alpha^\alpha.$$

Applications

- *(moving maxima)* Let $f \in L_+^\alpha(dx)$ and $\mu(dx) = dx$. Then, the process

$$X_t := \int_{\mathbb{R}}^{e} f(t+x) M_\alpha(dx), \quad -\infty < t < \infty,$$

is mixing. Indeed,

$$d_\alpha(\tau) = \int_{\mathbb{R}} f^\alpha(\tau + x) \wedge f^\alpha(x) dx \longrightarrow 0, \quad \text{as } \tau \to \infty.$$

- *(mixed moving maxima)* (M3 processes of Smith and Weissman)

For $f(t,v) \in L_+^\alpha(dx \times \nu(dv))$ and $\mu(dt, dv) = dt \times \nu(dv)$, the process

$$X_t := \int_{\mathbb{R}\times V}^{e} f(t+x, v) M_\alpha(dx, dv), \quad -\infty < t < \infty,$$

is mixing.

- *(non–ergodicity)* Let $(E, \mathcal{E}, \mu)$ be $([0, 2\pi), \mathcal{B}, dx)$ and

$$X_t = \int_{[0,2\pi)}^{e} \sin^2(t+x) M_\alpha(dx), \quad -\infty < t < \infty.$$

The process $X = \{X_t\}_{t\in\mathbb{R}}$ is strictly stationary and non–ergodic.

Applications (cont'd)

• Indeed, with $g(x) = f(x) = \sin(x)$ for any $p > 0$,

$$\|U_\tau g \wedge g\|^p_{L^\alpha(dx)} = (\int_0^{2\pi} |\sin(\tau + x) \wedge \sin(x)|^{2\alpha} dx)^{p/\alpha} = \varphi(\tau),$$

is a non–zero $2\pi-$periodic function and so

$$\frac{1}{T}\int_0^T \varphi(\tau)d\tau \not\longrightarrow 0, \quad \text{as } T \to \infty.$$

• Hence, the necessary and sufficient condition for ergodicity

$$\frac{1}{T}\int_0^T \|U_\tau g \wedge g\|^p_{L^\alpha(dx)} d\tau \longrightarrow 0, \quad \text{as } T \to \infty$$

is violated.

Doubly stochastic processes: An example of Brown and Resnick

Let $(E, \mathcal{E}, \mu)$ be a *probability space* and $w = \{w_t\}_{t\geq 0}$ be a standard Brownian motion there.

• Define the *doubly stochastic process*

$$X_t = {}^{e}\!\!\int_E e^{w_t(u)-\alpha t/2} M_\alpha(du), \quad 0 \leq t < \infty,$$

where $M_\alpha,\ \alpha > 0$ is defined on a different probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

• Brown and Resnick (1977) introduced $X = \{X_t\}_{t\geq 0}$ and showed that it is stationary.

Prop [S. (2007)] X is exponentially mixing. Indeed, with $\alpha = 1$,

$$\begin{aligned} d_\alpha(\tau) &= \int_E (e^{w_\tau(u)-\tau/2} \wedge e^{w_0(u)-0/2})\mu(du) = \mathbb{E}_\mu(e^{w_\tau-\tau/2} \wedge 1) \\ &\leq \mu\{w_\tau > \tau(1/2-\epsilon)\} + \mathbb{E}_\mu e^{w_\tau-\tau/2} 1_{w_\tau \leq \tau(1/2-\epsilon)} \leq \Phi(-\sqrt{\tau}(1/2-\epsilon)) + e^{-\tau\epsilon}. \end{aligned}$$

Statistical applications

Let $\{X_t\}_{t\in\mathbb{R}}$ be stationary and ergodic.

Problem: Estimate the dependence function

$$d_\alpha(\tau) := d_\alpha(X_\tau, X_0).$$

- A consistent estimator of the 'dependence function' is:

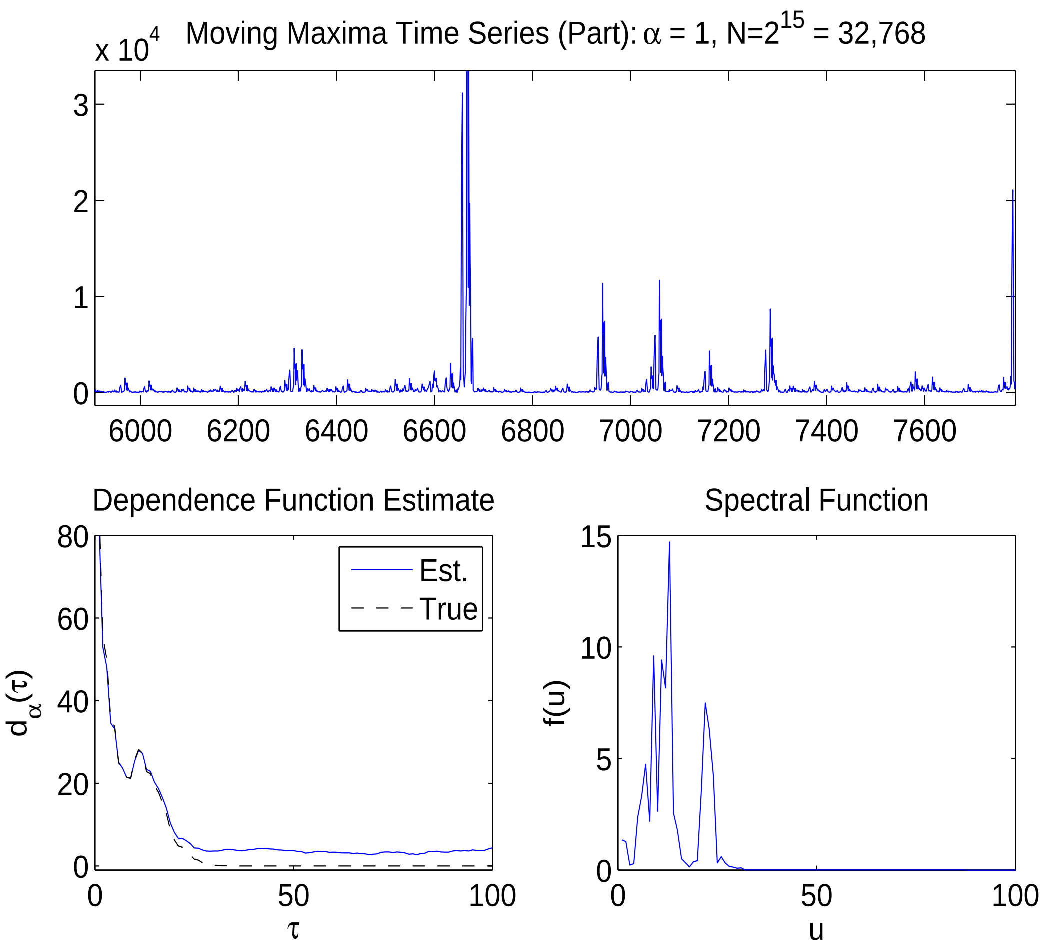
$$\widehat{d}_{\alpha,p,n}(\tau) := c_{p,\alpha}\Big(\frac{2}{n}\sum_{k=1}^{n} X_k^p\Big)^{\alpha/p} - c_{p,\alpha}\Big(\frac{1}{n-\tau}\sum_{k=1}^{n-\tau}(X_{\tau+k}\vee X_k)^p\Big)^{\alpha/p},$$

where $0 < p < \alpha$ and $c_{p,\alpha} = \Gamma(1-p/\alpha)^{-\alpha/p}$.

- By ergodicity, for all $\gamma \in (0,1)$

$$\widehat{d}_{\alpha,p,n}(\tau) \stackrel{a.s.}{\to} d_\alpha(\tau) \quad \text{and} \quad \mathbb{E}|\widehat{d}_{\alpha,p,n}(\tau) - d_\alpha(\tau)|^\gamma \to 0, \quad \text{as } n \to \infty.$$

A numerical example



Concluding remarks

• Weintraub (1991) introduced 3 notions of “mixing” in terms of the de Haan’s spectral representation.

◦ No connections with the classical notions of ergodicity and mixing was established.

• We show that Weintraub’s ’0–mixing’ is equivalent to mixing.

• Our work justifies and suggests a range of statistical methods for max–stable processes.

• Some old new tools on modeling and statistics for max–stable processes/fields?

• Further questions on:

◦ estimation of the spectral function

◦ representations of max–stable processes

◦ random fields

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One idea behind the proofs

Let $A \in \sigma\{X_t, -\infty < t \le 0\}$ and $B \in \sigma\{X_t, 0 < t < \infty\}$.

- Can be approximated with events of the type:

$$A = \{X_{s_i} \le x_i,\ 1 \le i \le d\} \quad \text{and} \quad B = \{X_{t_i} \le y_i,\ 1 \le i \le d\},$$

with $s_i \le 0$ and $t_i > 0$, $x_i, y_i \ge 0$.

- Note that

$$\mathbb{P}(A) = \mathbb{P}\{\vee_{i=1}^d x_i^{-1} X_{s_i} \le 1\} = \exp\{-\int_E \vee_{i=1}^d x_i^{-\alpha} f_{s_i}^\alpha(u)\mu(du)\}$$

and similarly

$$\mathbb{P}(A \cap B_\tau) = \exp\{-\int_E g^\alpha(u) \vee U_\tau(h)^\alpha(u)\mu(du)\},$$

where $g(u) = \vee_{i=1}^d x_i^{-1} f_{s_i}(u)$ and $h(u) = \vee_{i=1}^d y_i^{-1} f_{t_i}(u)$.

- Thus $\mathbb{P}(A \cap B_\tau)/\mathbb{P}(A)\mathbb{P}(B) \to 1$ as $\tau \to \infty$ if and only if

$$\exp\{\int_E g^\alpha d\mu + \int_E h^\alpha d\mu - \int_E g^\alpha \vee U_\tau(h)^\alpha d\mu\} \longrightarrow 1, \quad \text{as } \tau \to \infty.$$

∘ that is, if and only if, $\int_E g^\alpha \wedge U_\tau(h)^\alpha d\mu \to 0$, as $\tau \to \infty$.

Thank you!