Extremal stochastic integrals: a parallel between max–stable processes and $\alpha-$stable processes

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- *Max–stable* distributions and processes
- *Fréchet processes*

Part II

- Max–stable random measures and extremal integrals
- de Haan's spectral representation

Part III

- Examples
- Connections to $\alpha-$ or sum–stable processes

Part I

- *Max–stable* distributions and processes

- *Fréchet processes*

Extreme value distributions

For i.i.d. $X_1, \ldots, X_n, \ldots$ consider the maxima

$$M_n := \max\{X_1, \ldots, X_n\} = X_1 \vee \cdots \vee X_n.$$

∘ When do the M_n's converge in distribution, under appropriate shift and rescaling? What are the possible limit distributions?

• Gnedenko (1943) completing earlier works of Fréchet (1927) and Fisher and Tippett (1928) showed that, if

$$\frac{1}{a_n} M_n - b_n \stackrel{d}{\longrightarrow} Z, \quad \text{as } n \to \infty,$$

then Z has either the Fréchet, the negative Fréchet or the Gumbel distribution, up to rescaling and shift.

• Namely, $\mathbb{P}\{Z \le x\}$ equals one of the following three c.d.f.'s

$$\begin{cases} \Phi_\alpha(x) & := \exp\{-x^{-\alpha}\} \quad , \ 0 < x < \infty, \\ \Psi_\alpha(x) & := \exp\{-(-x)^{\alpha}\} \quad , \ -\infty < x < 0, \\ \Lambda(x) & := \exp\{-e^{-x}\} \quad , \ -\infty < x < \infty, \end{cases}$$

for some $\alpha > 0$.

• For characterizations of the domains of max–attraction of these three *extreme value distributions*, see e.g. Ch. 1 in Resnick (1986).

Max–stable distributions

Def *A random variable X has a max–stable distribution, if $\forall\ a,\ b > 0$, $\exists c > 0,\ d \in \mathbb{R}$, such that*

$$aX' \vee bX'' \stackrel{d}{=} cX + d,$$

where X' and X'' are independent copies of X.

• The extreme value distributions are also max–stable and vice versa.

◦ What happens in the multivariate situation?

Def *A random vector $\mathbf{X} = (X_1 \cdots X_k)^t$ taking values in $\mathbb{R}^k$ is max–stable, if $\forall\, \mathbf{a},\ \mathbf{b} \in \mathbb{R}^k_+$, $\exists\, \mathbf{c} \in \mathbb{R}^k_+$ and $\mathbf{d} \in \mathbb{R}^k$, such that*

$$\mathbf{a}\mathbf{X}' \vee \mathbf{b}\mathbf{X}'' \stackrel{d}{=} \mathbf{c}\mathbf{X} + \mathbf{d},$$

where $\mathbf{X}'$ and $\mathbf{X}''$ are independent copies of $\mathbf{X}$ and where the multiplication, the addition and the maxima operations are taken component–wise.

• The marginals of a multivariate max–stable distribution are max–stable.

◦ What is the dependence structure? Are they also *extreme value distributions*?

◦ These and other questions concerning multivariate max–infinite divisibility have been resolved (see, e.g. Ch. 5 in Resnick (1986)).

• We next take an alternative route, which will lead us to a constructive characterization of the multivariate max–stable laws.

Fréchet variables and processes

A random variable Z is $\alpha-$Fréchet, $\alpha > 0$, if

$$\mathbb{P}\{Z \le x\} = \text{exp}\{-\sigma^\alpha x^{-\alpha}\}, \quad \text{for } x > 0,$$

where $\sigma \ge 0$ is called the *scale coefficient* of Z.

If $\sigma = 0$, then $Z = 0$ is trivial.

• *max–stability:* For all $a, b > 0$, and iid $Z,\ Z', Z''$,

$$\begin{aligned}\mathbb{P}\{aZ' \vee bZ'' \le x\} &= \mathbb{P}\{Z \le x/a\}\mathbb{P}\{Z \le x/b\} \\ &= \text{exp}\{-\sigma^\alpha(a^\alpha + b^\alpha)x^{-\alpha}\}.\end{aligned}$$

• *scale–invariance:* For all $a > 0$, aZ is $\alpha-$Fréchet with scale coefficient $a\sigma$.

◦ In particular, for iid $Z, Z_1, \ldots, Z_n$,

$$Z_1 \vee \cdots \vee Z_n \stackrel{d}{=} n^{1/\alpha} Z.$$

• *heavy tails:* For $\sigma > 0$, as $x \to \infty$,

$$\mathbb{P}\{Z > x\} = 1 - \text{exp}\{-\sigma^\alpha x^{-\alpha}\} \sim \sigma^\alpha x^{-\alpha}.$$

Def *A stochastic process $X = \{X_t\}_{t\in T}$ is an $\alpha-$Fréchet process, $\alpha > 0$, if $\forall t_i \in T$, and $a_i \ge 0$, $i = 1, \ldots, n$, the max–linear combination*

$$Z = a_1 X_{t_1} \vee \cdots \vee a_n X_{t_n}$$

is an $\alpha-$Fréchet random variable.

Frechet and max–stable processes

• Let $Z_i,\ i \in \mathbb{N}$ be iid $\alpha-$Fréchet variables and let

$$X_t := \bigvee_{i=1}^{n_t} a_{t,i} Z_i, \quad t \in T,\ a_{t,i} \geq 0,\ i = 1, \ldots, n_t,$$

$n_t \in \mathbb{N}$.

Then $X = \{X_t\}_{t \in T}$ is an $\alpha-$Fréchet process.

∘ Why are Fréchet processes interesting?

Def *A process $X = \{X_t\}_{t \in T}$ with max–stable f.d.d. is called a max–stable process.*

• Any $\alpha-$Fréchet process is a max–stable process.

• Conversely, any max–stable process X with $\alpha-$Fréchet marginals is an $\alpha-$Fréchet process (de Haan (1978)).

• Thus the dependence structure of a max–stable process is fully captured by the class of Fréchet processes.

∘ How can we construct and handle $\alpha-$Fréchet processes?

• *Wold–type device*: For all $x_i > 0,\ i = 1, \ldots, k$,

$$\mathbb{P}\{X_{t_1} \leq x_1, \cdots, X_{t_k} \leq x_k\} = \mathbb{P}\{\bigvee_{i=1}^{k} x_i^{-1} X_{t_i} \leq 1\}.$$

Thus, the behavior of the max–linear combinations $\bigvee_i a_i X_{t_i}$ determines the f.d.d. of the process X.

Part II

- Max–stable random measures and extremal integrals

- de Haan's spectral representation

Fréchet random sup–measures

Let $(E, \mathcal{E}, \mu)$ be a measure space, with μ positive and $\sigma-$additive. Let also

$$\mathcal{E}_0 = \{A \in \mathcal{E} \,:\, \mu(A) < \infty\}, \quad \text{and} \quad \alpha > 0.$$

Def *The map $M_\alpha : \mathcal{E}_0 \to \mathcal{L}^0(\Omega)$ is an $\alpha-$Fréchet random sup–measure, if:*

(i) $M_\alpha(A_1), \ldots, M_\alpha(A_n)$ are independent for disjoint A_i's.

(ii) $\mathbb{P}\{M_\alpha(A) \le x\} = \exp\{-\mu(A)x^{-\alpha}\},\ x > 0,\ A \in \mathcal{E}_0$.

(iii) If $\cup_{i\in\mathbb{N}} A_i \in \mathcal{E}_0$, then

$$M_\alpha(\bigcup_{i\in\mathbb{N}} A_i) = \sup_{i\in\mathbb{N}} M_\alpha(A_i) \equiv \bigvee_{i\in\mathbb{N}} M_\alpha(A_i).$$

The measure μ is called the control measure of M_α.

- For $(E, \mathcal{E}, \mu) \equiv (\mathbb{R}, \mathcal{B}, |\cdot|)$, M_α assigns independent $\alpha-$Fréchet variables to disjoint intervals. The larger the interval, the larger the scale coefficient of its random measure.

- The measure of a disjoint union is the *maximum* of the measures, and not their sum!

$$M_\alpha((a,b) \cup (c,d)) = M_\alpha(a,b) \vee M_\alpha(c,d), \quad (a,b) \cup (c,d) = \emptyset.$$

- Sup–measures in general setting have been studied before, e.g. Vervaat (1997) and Resnick and Roy (1991).

- Here the focus is on the $\alpha-$Fréchet case.

Fréchet sup–measures and extremal integrals

• Fréchet random sup–measures exist (Stoev and Taqqu (2006)).

∘ How to define an integral with respect to M_α?

• Let $f(x) = \sum_{i=1}^n a_i 1_{A_i}(x)$, with disjoint A_i's and $a_i \geq 0$.

Def The *extremal integral* of $f(x)$ w.r.t. M_α is

$$\int_E^e f(x) M_\alpha(dx) := \bigvee_{i=1}^n a_i \int_{A_i}^e M_\alpha(dx) = \bigvee_{i=1}^n a_i M_\alpha(A_i).$$

Properties

(i) *(max–linearity)* For any simple $f(x),\ g(x) \geq 0$, and $a,\ b \geq 0$,

$$\int_E^e af(x) \vee bg(x) M_\alpha(dx) = a \int_E^e f(x) M_\alpha(dx) \vee b \int_E^e g(x) M_\alpha(dx).$$

(ii) *(scale coefficient isometry)* For any simple $f(x) \geq 0$, $\int_E^e f(x) M_\alpha(dx)$ is an $\alpha-$Fréchet variable with scale coefficient σ, such that

$$\sigma^\alpha =: \Big\| \int_E^e f(u) M_\alpha(du) \Big\|_\alpha^\alpha = \int_E f(u)^\alpha \mu(du).$$

Notation: The scale coefficient σ of an $\alpha-$Fréchet variable Z, is denoted by

$\|Z\|_\alpha := \sigma$ that is $\mathbb{P}\{Z \leq x\} = \exp\{-\|Z\|_\alpha^\alpha x^{-\alpha}\},\ x > 0.$

Extremal stochastic integrals (cont'd)

(iii) *(independence)* ${}^{e}\!\!\int_E f(x)M_\alpha(dx)$ and ${}^{e}\!\!\int_E g(x)M_\alpha(dx)$ are independent, iff f and g have disjoint supports i.e. $f(x)g(x) = 0$, $\mu-$a.e.

(iv) For simple $f_t(x) \geq 0$, the process $X = \{X_t\}_{t\in T}$,

$$X_t := {}^{e}\!\!\int_E f_t(x)M_\alpha(dx),\ t \in T,$$

is $\alpha-$Fréchet.

Moreover, by max–linearity, $\forall a_i \geq 0$,

$$\left\| \bigvee_{i=1}^{n} a_i X_{t_i} \right\|_\alpha^\alpha = \int_E \left(\bigvee_{i=1}^{n} a_i f_{t_i}(x) \right)^\alpha \mu(dx).$$

• Thus, for all $x_i > 0$,

$$\mathbb{P}\{X_{t_i} \leq x_i,\ i = 1, \ldots, n\} = \exp\left\{ -\int_E \left(\bigvee_{i=1}^{n} x_i^{-\alpha} f_{t_i}^\alpha(u) \right) \mu(du) \right\}.$$

• The f.d.d. of the process X can be expressed through the kernels of the extremal integrals.

∘ Can extremal integrals be defined for other than simple functions?

∘ Can they be used to construct an arbitrary max–stable law with $\alpha-$Fréchet marginals?

Extremal stochastic integrals of general integrands

Def (i) *A set $\mathcal{C}$ of r.v. is jointly $\alpha-$Fréchet, if max–linear combinations are $\alpha-$Fréchet.* **(ii)** *It is called an $\alpha-$Fréchet space if it is also closed w.r.t. max–linear combinations.*

Th *(Stoev and Taqqu (2006)) For an $\alpha-$Fréchet space $\mathcal{C}$, and $\xi_n,\ \xi \in \mathcal{C}$ the following are equivalent:*

(i) $\xi_n \to^P \xi$, *as* $n \to \infty$.

(ii) $\rho_{\alpha,\mathcal{C}}(\xi_n, \xi) := 2\|\xi_n \vee \xi\|_\alpha^\alpha - \|\xi_n\|_\alpha^\alpha - \|\xi\|_\alpha^\alpha \to 0$, *as* $n \to \infty$.

(iii) $m_p(\xi_n, \xi) = \mathbb{E}|\xi_n^p - \xi^p| \to 0$, *as* $n \to \infty$, *with any* $p \in (0, \alpha)$.

Moreover, $\rho_{\alpha,\mathcal{C}}$ *and* m_p *are metrics.*

- The set $\mathcal{C}_0 := \{{}^e\!\!\int_E f(u) M_\alpha(du),\ f$ − simple$\}$ is an $\alpha-$Fréchet space.
- The plim–closure of $\mathcal{C}_0$ is an $\alpha-$Fréchet space.
- For simple f_n's, let $\xi_n := {}^e\!\!\int_E f_n(x) M_\alpha(dx)$. We have

$$\rho_{\alpha,\mathcal{C}_0}(\xi_n, \xi_m) = \int_E |f_n(x)^\alpha - f_m^\alpha(x)| \mu(dx).$$

Thus, for any $f \in L_+^\alpha(\mu)$, one can define

$${}^e\!\!\int_E f(x) M_\alpha(dx) := \mathrm{plim}_{n\to\infty} {}^e\!\!\int_E f_n(x) M_\alpha(dx)$$

where $\int_E |f_n^\alpha(x) - f^\alpha(x)|\mu(dx) \to 0,\ n \to \infty$, with suitable simple f_n's.

de Haan's spectral representation

• Multivariate max–stable laws have been characterized.

• Let $\mathbf{X} = (X_1 \cdots X_k)^t$ be max–stable with $\alpha-$Fréchet marginals, for some $\alpha > 0$. Then, there exist $f_i(x) \in L^1(dx, [0,1])$, $i = 1, \ldots, k$, such that

$$\mathbb{P}\{X_i \le x_i,\ i = 1, \ldots, k\} = \exp\left\{ -\int_0^1 \left(\bigvee_{i=1}^{k} x_i^{-\alpha} f_i(u) \right) du \right\},$$

for any $x_i > 0$, $i = 1, \ldots, x_k$ (see, e.g. Proposition 5.11 in Resnick (1986)).

• Thus, any multivariate max–stable law with $\alpha-$Fréchet marginals can be represented in terms of extremal integrals.

• de Haan (1984) also showed that a continuous in probability max–stable process $X = \{X_t\}_{t\in\mathbb{R}}$, has the representation:

$$X_t = \bigvee_{i\in\mathbb{N}} f_t(U_i)\Gamma_i^{-1/\alpha},$$

where $\Gamma_i = \epsilon_1 + \cdots + \epsilon_i$ are the arrivals of a homogeneous Poisson process, and U_i are i.i.d. taking values in $(0, 1)$, independent of the Γ_i's.

This is with appropriate functions $f_t(u) \in L^\alpha_+(\rho(du))$, where $\rho(du)$ is the probability distribution of the U_i's.

de Haan's spectral representation (cont'd)

• de Haan (1984) also introduced the integral functional

$$\xi(f) := {}^{\vee}\!\!\int_{[0,1]} f \equiv \bigvee_{i\in\mathbb{N}} f(U_i)\Gamma_i^{-1/\alpha},$$

defined for any $f(u) \in L^\alpha(\rho(du))$.

• $\xi(f)$ has similar properties to our extremal integrals.

• In fact,

$$\{\xi(f)\}_{f\in L^1_+(\rho)} \stackrel{d}{=} \{ {}^{e}\!\!\int_{[0,1]} f(u) M_\alpha(du)\}_{f\in L^1_+(\rho)},$$

for an $\alpha-$Fréchet sup–measure M_α on $([0,1],\mathcal{B})$ with control measure ρ.

• One can use either de Haan's spectral representation or extremal integrals to handle max–stable processes.

◦ Why use extremal integrals?

Part III

- Examples

- Connections to $\alpha-$ or sum–stable processes

Why use extremal integrals?

Let M_α be an $\alpha-$Fréchet sup–measure on $(E, \mathcal{E}, \mu)$.

• *(moving maxima)* For a kernel $f \in L^\alpha_+(dx)$, define

$$X_t := {\int_{\mathbb{R}}}^{e} f(t-x) M_\alpha(dx), \quad t \in \mathbb{R}.$$

Here $(E, \mathcal{E}, \mu) \equiv (\mathbb{R}, \mathcal{B}, dx)$.

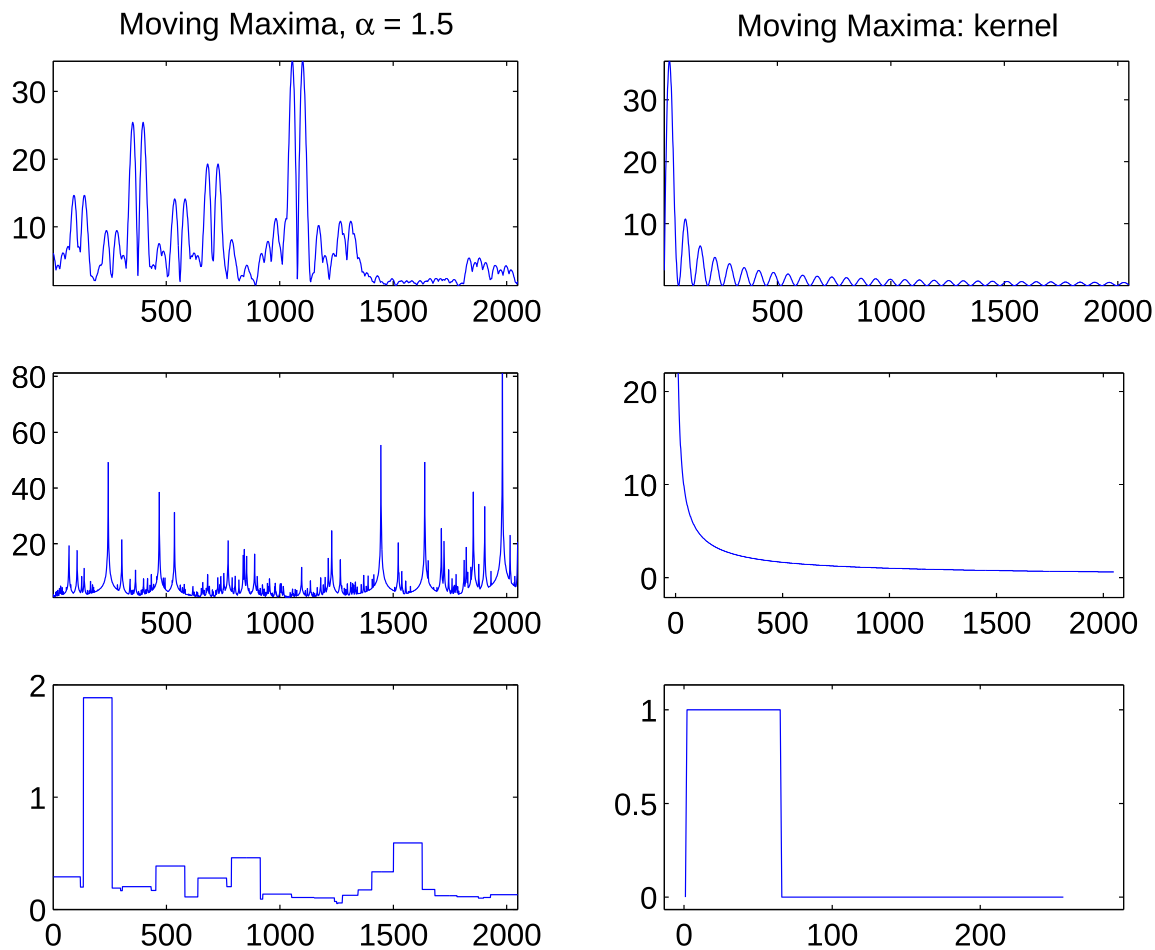
• It is hard to define these processes explicitly through the usual de Haan's representation (de Haan and Pickands (1986)).

• *(mixed moving maxima)* For a kernel $f \in L^\alpha_+(dx, \nu(dy))$, define

$$X_t := {\int_{\mathbb{R}\times A}}^{e} f(t-x, dy) M_\alpha(dx, dy), \quad t \in \mathbb{R}.$$

Here $(E, \mathcal{E}, \mu) \equiv (\mathbb{R} \times A, \mathcal{B} \times \mathcal{A}, dx \times \nu(dy))$.

Some examples



Examples of $\alpha-$Fréchet moving maxima.

- Wide variety of models can be obtained.

Sum–stable processes

- X is (sum–)stable, if $\forall a, b > 0$, $\exists c > 0, d$, such that

$$aX' + bX'' \overset{d}{=} cX + d,$$

for i.i.d. X, X', X''.

- symmetric stable laws have characteristic functions

$$\mathbb{E}e^{i\theta X} = \exp\{-\sigma^\alpha |\theta|^\alpha\}, \quad 0 < \alpha \le 2,\ \theta \in \mathbb{R},$$

α is index of stability, σ – scale coefficient.

- If $\alpha = 2$, then X is Gaussian.

- *(heavy tails)* If $0 < \alpha < 2$, then

$$\mathbb{P}\{X > x\} \sim Cx^{-\alpha}, \quad \text{as } x \to \infty.$$

Hence, for $p > 0$,

$$\mathbb{E}|X|^p < \infty \quad \text{iff} \quad 0 < p < \alpha.$$

◦ We focus on symmetric α–stable (SαS) distributions.

Def *$X = \{X_t\}_{t \in T}$ is a SαS process if all its finite linear combinations: $a_1 X_{t_1} + \cdots + a_n X_{t_n}$ are SαS.*

◦ How to handle SαS processes?

Sum–stable processes (cont'd)

• Let $\Lambda = \{\Lambda_t\}_{t\in\mathbb{R}}$ be a Lévy symmetric $\alpha-$stable process: Λ has *independent* and *stationary increments* with

$$\mathbb{E}e^{i\theta\Lambda_t} = \exp\{-|t||\theta|^\alpha\}, \quad \theta \in \mathbb{R}.$$

Λ is like Brownian motion in infinite variance and

$$X_t := \int_{\mathbb{R}} f_t(x) d\Lambda_t(x), \quad t \in T,$$

can be defined for all $f_t(x) \in L^\alpha(dx)$.

• This yields rich classes of SαS processes $X = \{X_t\}_{t\in T}$ (Samorodnitsky and Taqqu (1994)).

∘ What is the connection with max–stable processes?

Domains of attraction of max–stable processes

Let $X^{(i)} = \{X_t^{(i)}\}_{t\in T}, i = 1, 2, \ldots$ be i.i.d. copies of the SαS process X with

$$0 < \alpha < 2.$$

• Let

$$M_n(t) := X_t^{(1)} \vee \cdots \vee X_t^{(n)}.$$

For fixed t, $\mathbb{P}\{X_t^{(i)} \geq x\} \sim Cx^{-\alpha},\ x \to \infty$, so

$$\frac{1}{n^{1/\alpha}} M_n(t) \stackrel{d}{\longrightarrow} Z, \quad \text{as } n \to \infty,$$

for an $\alpha-$Fréchet Z.

◦ What happens with the f.d.d.'s?

Th*(Stoev and Taqqu (2006)) As $n \to \infty$,*

$$\{\frac{1}{n^{1/\alpha}} M_n(t)\}_{t\in T} \stackrel{f.d.d.}{\longrightarrow} \text{const}\{Z_t\}_{t\in T},$$

where

$$Z_t = {\int_{\mathbb{R}}}^{e} f_{t,+}(x) M'_\alpha(dx) \vee {\int_{\mathbb{R}}}^{e} f_{t,-}(x) M''_\alpha(dx),$$

for independent $\alpha-$Fréchet measures M'_α and M''_α on $(\mathbb{R}, \mathcal{B}, dx)$.

Here

$$f_{t,+}(x) = \max\{f_t(x), 0\} \quad \textit{and} \quad f_{t,-}(x) = \max\{-f_t(x), 0\}.$$

Domains of attraction (cont'd)

Idea of proof: For any $a_j > 0$,

$$\bigvee_{j=1}^{k} a_j M_n(t_j) = \bigvee_{i=1}^{n} \xi_i,$$

where

$$\xi_i = \bigvee_{j=1}^{k} a_j X_{t_i}^{(i)}.$$

- Need to examine the tail behavior of ξ_i's.
- By Th 4.4.5 in Samorodnitsky and Taqqu (1994):

$$\mathbb{P}\{\xi_1 \geq x\} \sim \text{const} \left(\int_{\mathbb{R}} (\vee a_j f_{t_j,+}(u))^\alpha du + \int_{\mathbb{R}} (\vee a_j f_{t_j,-}(u))^\alpha du \right) x^{-\alpha}.$$

- The Wold–type device yields the result. □

∘ Essentially any $\alpha-$stable process belongs to the domain of max–attraction of an $\alpha-$Fréchet process.

∘ The dependence structure of the max–stable limits can be very rich.

∘ The f.d.d. of the max–stable limits are characterized explicitly.

Summary

Extremal integrals provide:

∘ alternative to de Haan's spectral representation

∘ natural tools to construct/model max–stable processes

∘ illuminate connections to $\alpha-$stable processes

Future work:

∘ "translate" more results from the $\alpha-$stable literature

∘ weak convergence in path spaces

∘ statistics for max–stable processes

Thank you!

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