

On the ergodicity and mixing of max-stable processes

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Outline

- ▶ Max-stable processes
- ▶ Representations
- ▶ Problem formulation
- ▶ Main results
- ▶ Examples & Applications

Max-stable processes

- $\{X_t\}_t$ is max-stable if all f.d.d. are max-stable. The marginals of X are thus either *Fréchet*, *Gumbel* or *negative Fréchet*.
- Recall that X is α -Fréchet ($\alpha > 0$) with scale $\sigma > 0$, if

$$\mathbb{P}\{X \leq x\} = \exp\{-(x/\sigma)^{-\alpha}\}, \quad x \in (0, \infty).$$

Def $\{X_t\}$ is an α -Fréchet process if for all $a_i > 0$ and t_i , $i = 1, \dots, d$, the *max-linear combination*

$$\max\{a_i X_{t_i}, 1 \leq i \leq d\} = \bigvee_{i=1}^d a_i X_{t_i},$$

is α -Fréchet.

Thm [de Haan (1978)] A process X with α -Fréchet marginals is max-stable *if and only if* it is an α -Fréchet process.

- The class of Fréchet processes is **sufficient** for the study of ergodicity.

Examples of max-stable processes ...

- ▶ **discrete moving maxima** For i.i.d. α -Fréchet Z_i 's,

$$X_t := \bigvee_{i=0}^{\infty} a_i Z_{t-i}, \quad t \in \mathbb{Z}$$

is well-defined stationary α -Fréchet process if $a_i \geq 0$ are such that

$$\sum_{i=0}^{\infty} a_i^\alpha < \infty.$$

- ▶ **maxima of moving maxima (M3)** For i.i.d. α -Fréchet $Z_i(j)$'s,

$$X_t := \max_j \bigvee_i a_i(j) Z_{t-i}(j), \quad t \in \mathbb{Z},$$

is well-defined stationary α -Fréchet process if $a_i(j) \geq 0$ and $\sum_j \sum_i a_i(j)^\alpha < \infty$.

Why ergodicity?

Let $\{X_k, k = 1, 2, \dots, n\}$ be a sample from a stationary max-stable process.

- Ergodicity implies that, if $\mathbb{E}|h(X_1)| < \infty$, then

$$\frac{1}{n} \sum_{k=1}^n h(X_k) \xrightarrow{\text{a.s.}} \mathbb{E}h(X_1), \quad \text{as } n \rightarrow \infty.$$

- Or, more generally, for $\mathbb{E}|h(X_1, \dots, X_m)| < \infty$, that

$$\frac{1}{n-m} \sum_{k=0}^{n-m-1} h(X_{k+1}, \dots, X_{k+m}) \xrightarrow{\text{a.s.}} \mathbb{E}h(X_1, \dots, X_m), \quad \text{as } n \rightarrow \infty.$$

- Thus ergodicity has important **statistical implications**.
- To characterize the ergodic properties of max-stable stationary processes one needs suitable **representations**.

The de Haan's spectral representation

Thm [the de Haan spectral representation] If $X = \{X_t\}_{t \in \mathbb{R}}$ is continuous in probability α -Fréchet process, then

$$\{X_t\} \stackrel{fdd}{=} \left\{ \bigvee_{k=1}^{\infty} f_t(U_k) / \epsilon_k^{1/\alpha} \right\},$$

for a Poisson point process $\{(U_k, \epsilon_k)\}$ in $(0, 1) \times (0, \infty)$ with intensity $\rho(du) \times dx$ and deterministic functions $f_t(\cdot) \geq 0$ with $\int_0^1 f_t(u)^\alpha \rho(du) < \infty$.

- Equivalently, one can use *extremal integrals*:

$$\{X_t\}_t \stackrel{fdd}{=} \left\{ \int_{(0,1)}^e f_t(u) M_\alpha(du) \right\}_t$$

where M_α is an α -Fréchet sup-measure on $(0, 1)$ with control measure $\rho(du)$.

- For convenience we use *extremal integrals* in the sequel.

Extremal integrals: intuition

- Observe that for the **moving maxima**, for example, for $x > 0$

$$\begin{aligned}\mathbb{P}\{X_t \leq x\} &= \prod_{i=0}^{\infty} e^{-(x/a_i)^{-\alpha}} = \exp\left\{-\sum_{i=0}^{\infty} a_i^\alpha x^{-\alpha}\right\} \\ &= \exp\left\{-\int_0^{\infty} f(u)^\alpha du x^{-\alpha}\right\},\end{aligned}$$

where $f(u) := \sum_{i=0}^{\infty} a_i 1_{[i, i+1)}(u)$, $0 \leq u < \infty$.

- One can write X_t as an **extremal integral**:

$$X_t = \int_{\mathbb{R}}^e f(t-u) M_\alpha(du) = \bigvee_{i=-\infty}^t f(t-i) M_\alpha(i-1, i],$$

where $M_\alpha(i-1, i] = Z_i$ are i.i.d. standard α -Fréchet variables, and

$$\int_{\mathbb{R}}^e \sum_i a_i 1_{A_i}(u) M_\alpha(du) := \bigvee_i a_i M_\alpha(A_i).$$

Random sup-measures

- This suggests defining continuous versions of **extremal integrals** with respect to a random sup-measure M_α .

Def M_α is α -Fréchet sup-measure on (E, \mathcal{E}) with control measure μ , if: (i) independently scattered (ii)

$\mathbb{P}\{M_\alpha(A) \leq x\} = \exp\{-\mu(A)x^{-\alpha}\}$, $x > 0$ (iii) σ -sup-additive:

$$M_\alpha(\cup_{n=1}^{\infty} A_n) = \bigvee_{n=1}^{\infty} M_\alpha(A_n), \quad \text{almost surely.}$$

- Analogous to the α -stable measures in the sum-stable case (see Samorodnitsky and Taqqu (1994)).

Intuition: Let $A \cap B = \emptyset$ and $a, b > 0$. By independence, for $x > 0$

$$\begin{aligned} \mathbb{P}\{aM_\alpha(A) \vee bM_\alpha(B) \leq x\} &= e^{-\mu(A)(x/a)^{-\alpha}} e^{-\mu(B)(x/b)^{-\alpha}} \\ &= e^{-(a^\alpha \mu(A) + b^\alpha \mu(B))x^{-\alpha}}. \end{aligned}$$

Extremal integrals

For $f(u) = \sum_{k=1}^n a_k 1_{A_k}(u)$, $a_k \geq 0$, define

$$I(f) := \int_E^e f(u) M_\alpha(du) := \bigvee_{k=1}^n a_k M_\alpha(A_k),$$

to be the **extremal integral** of f w.r.t. M_α .

Properties:

1. For simple $f(u) \geq 0$,

$$\mathbb{P}\{I(f) \leq x\} = e^{-\int_E f^\alpha d\mu \cdot x^{-\alpha}}, \quad x > 0.$$

2. For constants $a, b > 0$, and simple $f(u), g(u) \leq 0$:

$$I(af \vee bg) = aI(f) \vee bI(g).$$

3. For simple $f_i(u) \geq 0$, $1 \leq i \leq d$, $I(f_i)$, $1 \leq i \leq d$ are independent iff f_i , $1 \leq i \leq d$ have disjoint supports (μ -a.e.).

Note: $\int_E f dM_\alpha$ can be defined for all measurable $f(u) \geq 0$, with $\int_E f^\alpha d\mu < \infty$.

Fréchet spaces

Notation: For α -Fréchet ξ , $\mathbb{P}\{\xi \leq x\} = \exp\{-cx^{-\alpha}\}$, set $\|\xi\|_\alpha := c^{1/\alpha}$, so that

$$\left\| \int_E f dM_\alpha \right\|_\alpha = \left(\int_E f^\alpha d\mu \right)^{1/\alpha}.$$

- The collection of all extremal integrals $I(f) = \int_E f dM_\alpha$, $f \in L_+^\alpha(\mu)$ is closed w.r.t. *max-linear combinations*:

$$a_1 I(f_1) \vee \cdots \vee a_n I(f_n) = I(a_1 f_1 \vee \cdots \vee a_n f_n), \quad \forall a_i \geq 0.$$

- They are *jointly Fréchet* or a *Fréchet space* (like Gaussian spaces).

A natural metric: For *Fréchet spaces*:

$$\rho_\alpha(\xi, \eta) = 2\|\xi \vee \eta\|_\alpha^\alpha - \|\xi\|_\alpha^\alpha - \|\eta\|_\alpha^\alpha,$$

is a metric, which metrizes the convergence in probability.

- For $\xi = \int_E f dM_\alpha$ and $\eta = \int_E g dM_\alpha$, $f, g \in L_+^\alpha(\mu)$

$$\rho_\alpha(\xi, \eta) = 2 \int_E f^\alpha \vee g^\alpha d\mu - \int_E f^\alpha d\mu - \int_E g^\alpha d\mu = \int_E |f^\alpha - g^\alpha| d\mu.$$

Pistons and max-linear isometries

Now for any $f_t \in L_+^\alpha(\mu)$

$$X_t = \int_E^e f_t(u) M_\alpha(du), \quad -\infty < t < \infty$$

is a max-stable process (all \vee -linear combinations are α -Fréchet).

Thm [de Haan and Pickands (1986)] If $X = \{X_t\}$ is stationary and continuous in probability, then one can take $E = [0, 1]$ and

$$f_t = \Gamma_t(f_0),$$

where $\{\Gamma_t\}$ is a group of **pistons**.

• The pistons are precisely the **max-linear isometries** of $L_+^\alpha([0, 1])$.

Def The map $U : L_+^\alpha(\mu) \rightarrow L_+^\alpha(\nu)$ is a max-linear isometry (m.l.i.) of measure spaces if: (i) $U(af \vee bg) = aU(f) \vee bU(g)$, for all $a, b \geq 0$ and $f, g \in L_+^\alpha(\mu)$ (ii) $\int_E f^\alpha d\mu = \int_F U(f)^\alpha d\nu$, for all $f \in L_+^\alpha(\mu)$.

Stationary max-stable processes

- Max-linear isometries $U : L_+^\alpha(\mu) \rightarrow L_+^\alpha(\mu)$ preserve the metric

$$\rho_\alpha(I(f), I(g)) = \rho_\alpha(I(Uf), I(Ug)),$$

where $\rho_\alpha(I(f), I(g)) = \int_E |f^\alpha - g^\alpha| d\mu$.

- For a group of m.l.i. $\{U_t\}_{t \in \mathbb{R}}$ on $L_+^\alpha(\mu)$ and $f_0 \in L_+^\alpha(\mu)$, the process

$$X_t := \int_E^e U_t(f_0) dM_\alpha, \quad t \in \mathbb{R}, \quad (1)$$

is a (strictly) stationary α -Fréchet process.

- We focus on stationary α -Fréchet processes as in (1), including, for example (but not limited to):
 - the moving maxima:

$$X_t := \int_{\mathbb{R}} f_0(t+u) M_\alpha(du), \quad t \in \mathbb{R}.$$

- ... or, more generally, the mixed moving maxima:

$$X_t := \int_{\mathbb{R} \times \mathcal{V}} f_0(t+u, v) M_\alpha(du, dv), \quad t \in \mathbb{R}.$$

Problem formulation

The process: Let $X = \{X_t\}_{t \in \mathbb{R}}$:

$$X_t := \int_E^e U_t(f) dM_\alpha, \quad -\infty < t < \infty,$$

where $f \in L_+^\alpha(\mu)$ and $\{U_t\}$ is a group of m.l.i. on $L_+^\alpha(\mu)$.

Ergodicity: we look for *necessary and sufficient conditions* (on $\{U_t\}$ and f) for

$$\xi_T(h) := \frac{1}{T} \int_0^T h \circ S_\tau(X) d\tau \xrightarrow{a.s.} \text{const}, \quad \text{as } T \rightarrow \infty,$$

for all bounded Borel measurable $h : \mathbb{R}^{\mathbb{R}} \rightarrow \mathbb{R}^{\mathbb{R}}$, where $S_\tau(\{x_t\}) = \{x_{t+\tau}\}$.

Mixing: we look for *necessary and sufficient conditions* for

$$\mathbb{P}(A \cap B_\tau) \longrightarrow \mathbb{P}(A)\mathbb{P}(B), \quad \text{as } \tau \rightarrow \infty,$$

for all $A \in \sigma\{X_t, -\infty < t \leq 0\}$ and $B \in \sigma\{X_t, 0 < t < \infty\}$, with $B_\tau = S_\tau(B)$.

Technicality: $\{X_t\}$ has a measurable modification if and only if it is continuous in probability (S. (2007)).

General Results: ergodicity

Thm [S. (2007)] $X = \{X_t\}_{t \in \mathbb{R}}$ is ergodic, if and only if, for some (any) $p > 0$,

$$\frac{1}{T} \int_0^T \|U_\tau g \wedge g\|_{L^\alpha(\mu)}^p d\tau \longrightarrow 0, \quad \text{as } T \rightarrow \infty,$$

for all $g \in F := \overline{\text{span}\{U_t f, t \in \mathbb{R}\}}$.

- Here $F \subset L_+^\alpha(\mu)$ contains all positive max-linear combinations of $U_t f$'s and is closed in the metric ρ_α .
- Namely, all limits in ρ_α of the max-linear combinations

$$a_1 U_{t_1}(f) \vee \cdots \vee a_n U_{t_n}(f), \quad \forall a_i \geq 0, t_i \in \mathbb{R}.$$

General Results: mixing

Thm [S. (2007)] $X = \{X_t\}_{t \in \mathbb{R}}$ is mixing, if and only if,

$$\|U_\tau h \wedge g\|_{L_+^\alpha(\mu)} \longrightarrow 0, \quad \text{as } \tau \rightarrow \infty,$$

for all $g \in F_-$ and $h \in F_+$, where

$$F_- = \overline{\text{span}\{U_t f, t \leq 0\}} \quad \text{and} \quad F_+ = \overline{\text{span}\{U_t f, 0 < t\}}.$$

- The proofs borrow ideas from the α -(sum)stable case –
Campanis, Hardin and Weron (1987).

Idea: $\|f \wedge g\|_{L_+^\alpha(\mu)}$ 'measures' the dependence b/w $\xi = \int_E f dM_\alpha$
and $\eta = \int_E g dM_\alpha$.

On the idea behind the proofs (mixing)

Any $A \in \sigma\{X_t, -\infty < t \leq 0\}$ and $B \in \sigma\{X_t, 0 < t < \infty\}$ can be approximated via events of the type:

$$A = \{X_{s_i} \leq x_i, 1 \leq i \leq d\} \quad \text{and} \quad B = \{X_{t_i} \leq y_i, 1 \leq i \leq d\},$$

with $s_i \leq 0$ and $t_i > 0$, $x_i, y_i \geq 0$.

- Note that

$$\mathbb{P}(A) = \mathbb{P}\{\bigvee_{i=1}^d x_i^{-1} X_{s_i} \leq 1\} = e^{-\int_E \bigvee_{i=1}^d x_i^{-\alpha} f_{s_i}^\alpha(u) \mu(du)}$$

and similarly $\mathbb{P}(A \cap B_\tau) = e^{-\int_E g^\alpha(u) \vee U_\tau(h)^\alpha(u) \mu(du)}$, where

$$g(u) = \bigvee_{i=1}^d x_i^{-1} f_{s_i}(u) \quad \text{and} \quad h(u) = \bigvee_{i=1}^d y_i^{-1} f_{t_i}(u).$$

- Thus $\mathbb{P}(A \cap B_\tau) / \mathbb{P}(A)\mathbb{P}(B) \rightarrow 1$ as $\tau \rightarrow \infty$ if and only if

$$e^{\int_E g^\alpha d\mu + \int_E h^\alpha d\mu - \int_E g^\alpha \vee U_\tau(h)^\alpha d\mu} = e^{\int_E g^\alpha \wedge U_\tau(h)^\alpha d\mu} \rightarrow 1 \quad (\tau \rightarrow \infty)$$

that is, if and only if, $\int_E g^\alpha \wedge U_\tau(h)^\alpha d\mu \rightarrow 0$, as $\tau \rightarrow \infty$.

A 'natural' measure of dependence

Def For jointly α -Fréchet ξ and η ,

$$d_\alpha(\xi, \eta) := \|\xi\|_\alpha^\alpha + \|\eta\|_\alpha^\alpha - \|\xi \vee \eta\|_\alpha^\alpha,$$

is a measure of the dependence between ξ and η .

- ξ and η are independent if and only if $d_\alpha(\xi, \eta) = 0$.
- If $\xi = \int_E f dM_\alpha$ and $\eta = \int_E g dM_\alpha$ then

$$d_\alpha(\xi, \eta) = \int_E f^\alpha d\mu + \int_E g^\alpha d\mu - \int_E f^\alpha \vee g^\alpha d\mu = \int_E f^\alpha \wedge g^\alpha d\mu$$

Thm [S. (2007)] A continuous in probability stationary α -Fréchet process $X = \{X_t\}_{t \in \mathbb{R}}$ is mixing if and only if $d_\alpha(X_t, X_0) \rightarrow 0$, as $t \rightarrow \infty$.

- The condition $d_\alpha(X_\tau, X_0) \rightarrow 0$, $\tau \rightarrow \infty$ is easy to check.
- One and the same process $X = \{X_t\}$ may have different representations $(f, \{U_t\}, M_\alpha)$, but has only one dependence function

$$d_\alpha(\tau) = d_\alpha(X_\tau, X_0) = \|X_\tau\|_\alpha^\alpha + \|X_0\|_\alpha^\alpha - \|X_\tau \vee X_0\|_\alpha^\alpha.$$

Applications

- (*moving maxima*) Let $f \in L^{\alpha}_{+}(dx)$ and $\mu(dx) = dx$. Then, the process

$$X_t := \int_{\mathbb{R}} f(t+x) M_{\alpha}(dx), \quad -\infty < t < \infty,$$

is *mixing*. Indeed,

$$d_{\alpha}(\tau) = \int_{\mathbb{R}} f^{\alpha}(\tau+x) \wedge f^{\alpha}(x) dx \longrightarrow 0, \quad \text{as } \tau \rightarrow \infty.$$

- (*mixed moving maxima*) (M3 processes of Smith and Weissman) For $f(t, v) \in L^{\alpha}_{+}(dx \times \nu(dv))$ and $\mu(dt, dv) = dt \times \nu(dv)$, the process

$$X_t := \int_{\mathbb{R} \times V} f(t+x, v) M_{\alpha}(dx, dv), \quad -\infty < t < \infty,$$

is *mixing*.

- (*non-ergodicity*) Let (E, \mathcal{E}, μ) be $([0, 2\pi), \mathcal{B}, dx)$ and

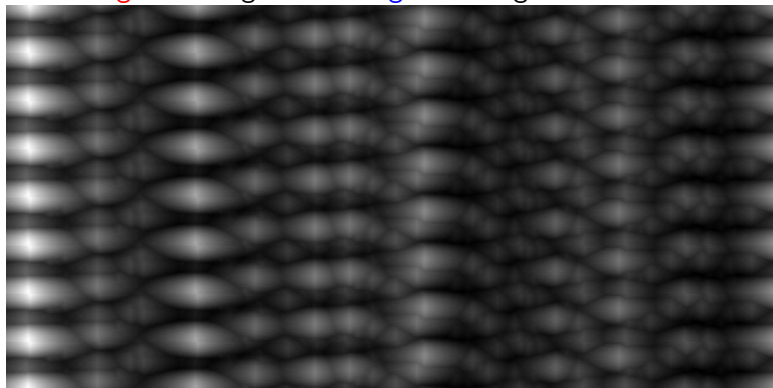
$$X_t = \int_{[0, 2\pi)} \sin^2(t+x) M_{\alpha}(dx), \quad -\infty < t < \infty.$$

An illustration

The stationary and non-isotropic field

$$X(s, t) = \int_{[0, \pi) \times \mathbb{R}}^e \sin^2(s + u) \times e^{-|t+v|} M_\alpha(du, dv)$$

is **non-ergodic** along “s” but **ergodic** along “t”.



A movie ...

Concluding remarks

- Weintraub (1991) introduced 3 notions of “mixing” in terms of the de Haan’s spectral representation.
 - No connections with the classical notions of ergodicity and mixing was established.
- We show that Weintraub’s ‘0–mixing’ is equivalent to mixing.
- Our work justifies and suggests a range of statistical methods for max–stable processes.
- Some old new tools on modeling and statistics for max–stable processes/fields?
- Further questions on:
 - estimation of the spectral function
 - representations of max–stable processes
 - random fields

Some references

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Appendix: α -Fréchet rv's and spaces

- $\|\xi\|_\alpha$ denotes the scale coefficient of an α -Fréchet variable:

$$\mathbb{P}\{\xi \leq x\} = e^{-\|\xi\|_\alpha^\alpha \cdot x^{-\alpha}}, \quad \forall x > 0.$$

- For an α -Fréchet variable ξ , and $0 < p < \alpha$:

$$\mathbb{P}\{\xi > x\} \sim \|\xi\|_\alpha^\alpha x^{-\alpha}, \quad (x \rightarrow \infty) \quad \text{and} \quad \mathbb{E}|\xi|^p = \|\xi\|_\alpha^p \Gamma(1 - \frac{p}{\alpha}).$$

Def A set \mathcal{C} of rv's is α -Fréchet if it: (i) contains α -Fréchet rv's on the same probability space; (ii) contains all max-linear combinations $\bigvee a_i \xi_i$, for all $a_i > 0$, $\xi_i \in \mathcal{C}$.

It is also *closed* if it contains its limits in probability.

- For an α -Fréchet space \mathcal{C} ,

$$\rho_\alpha(\xi, \eta) := 2\|\xi \vee \eta\|_\alpha^\alpha - \|\xi\|_\alpha^\alpha - \|\eta\|_\alpha^\alpha, \quad \xi, \eta \in \mathcal{C}$$

is a metric, which metrizes the convergence in probability.

- A closed α -Fréchet space \mathcal{C} is a complete metric space w.r.t. ρ_α .

Appendix: An example of Brown and Resnick

Let (E, \mathcal{E}, μ) be a *probability space* and $w = \{w_t\}_{t \geq 0}$ be a standard Brownian motion there.

- Define the *doubly stochastic process*

$$X_t = \int_E e^{w_t(u) - \alpha t/2} M_\alpha(du), \quad 0 \leq t < \infty,$$

where M_α , $\alpha > 0$ is defined on a different probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- Brown and Resnick (1977) introduced $X = \{X_t\}_{t \geq 0}$ and showed that it is stationary.

Prop [S. (2007)] X is exponentially mixing. Indeed, with $\alpha = 1$,

$$\begin{aligned} d_\alpha(\tau) &= \int_E (e^{w_\tau(u) - \tau/2} \wedge e^{w_0(u) - 0/2}) \mu(du) = \mathbb{E}_\mu(e^{w_\tau - \tau/2} \wedge 1) \\ &\leq \mu\{w_\tau > \tau(1/2 - \epsilon)\} + \mathbb{E}_\mu e^{w_\tau - \tau/2} 1_{w_\tau \leq \tau(1/2 - \epsilon)} \\ &\leq \Phi(-\sqrt{\tau}(1/2 - \epsilon)) + e^{-\tau\epsilon}. \end{aligned}$$

Appendix: Statistical applications

Let $\{X_t\}_{t \in \mathbb{R}}$ be **stationary** and **ergodic**.

Problem: Estimate the dependence function

$$d_\alpha(\tau) := d_\alpha(X_\tau, X_0).$$

- A consistent estimator of the 'dependence function' is:

$$\widehat{d}_{\alpha,p,n}(\tau) := c_{p,\alpha} \left(\frac{2}{n} \sum_{k=1}^n X_k^p \right)^{\alpha/p} - c_{p,\alpha} \left(\frac{1}{n-\tau} \sum_{k=1}^{n-\tau} (X_{\tau+k} \vee X_k)^p \right)^{\alpha/p},$$

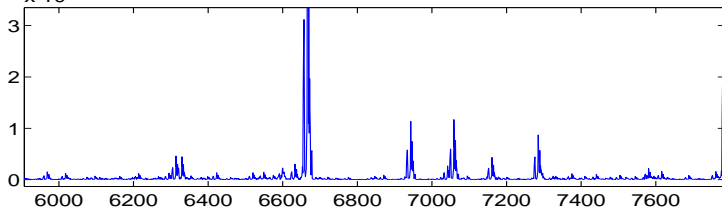
where $0 < p < \alpha$ and $c_{p,\alpha} = \Gamma(1 - p/\alpha)^{-\alpha/p}$.

- By ergodicity, for all $\gamma \in (0, 1)$

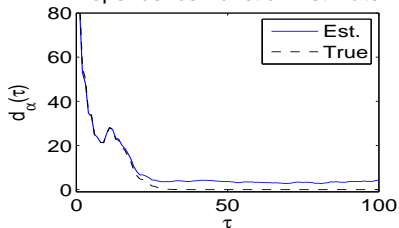
$$\widehat{d}_{\alpha,p,n}(\tau) \xrightarrow{\text{a.s.}} d_\alpha(\tau) \quad \text{and} \quad \mathbb{E}|\widehat{d}_{\alpha,p,n}(\tau) - d_\alpha(\tau)|^\gamma \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

Appending: A numerical example

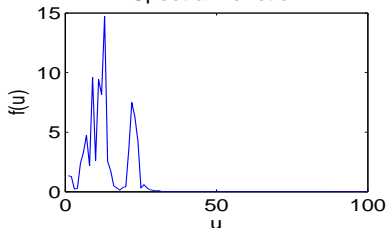
$\times 10^4$ Moving Maxima Time Series (Part): $\alpha = 1$, $N=2^{15} = 32,768$



Dependence Function Estimate



Spectral Function



Thank you!