

Max-stable processes: representations, ergodic properties and some statistical applications

Stilian Stoev
University of Michigan, Ann Arbor

Oberwolfach
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The focus

Let $X = \{X_t\}_{t \in \mathbb{R}}$ be a (strictly) stationary max-stable process.

Questions:

- When is X ergodic or mixing?
- How can one check ergodicity/mixing for X ?

Answers: Must be in terms of representations for max-stable processes.

- 1 The focus
- 2 Representations of max-stable processes
- 3 The problems of ergodicity and mixing
- 4 Results
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Max-stable processes

The process $X = \{X_t\}_{t \in \mathbb{R}}$ is a max-stable process if its finite-dimensional distributions (fdd) are multivariate max-stable:

Definition

A vector $\mathbf{X} = \{X_j\}_{1 \leq j \leq d}$ is max-stable if for all $k \in \mathbb{N}$, there exist $\mathbf{a}_k \in \mathbb{R}_+^d$, $\mathbf{b}_k \in \mathbb{R}^d$, such that

$$\frac{1}{\mathbf{a}_k} \bigvee_{i=1}^k \mathbf{X}^{(i)} - \mathbf{b}_k \stackrel{d}{=} \mathbf{X},$$

where $\mathbf{X}^{(i)}$, $1 \leq i \leq k$ are independent copies of \mathbf{X} .

- The marginals of X are either *Fréchet*, *Gumbel* or *negative Fréchet*.

Fréchet processes

- ξ is an α -Fréchet variable ($\alpha > 0$) with scale $\sigma > 0$, if

$$\mathbb{P}\{\xi \leq x\} = \exp\{-(x/\sigma)^{-\alpha}\} = \exp\{-\sigma^\alpha x^{-\alpha}\}, \quad x \in (0, \infty).$$

- $\mathbb{P}\{\xi > x\} \sim \sigma^\alpha x^{-\alpha}$, as $x \rightarrow \infty$.

Definition

$\{X_t\}_{t \in \mathbb{R}}$ is an α -Fréchet process if for all $a_i > 0$ and t_i , $1 \leq i \leq d$, the *max-linear combinations*

$$\max\{a_i X_{t_i}, 1 \leq i \leq d\} = \bigvee_{i=1}^d a_i X_{t_i},$$

are α -Fréchet.

Examples

- (moving maxima) For iid α -Fréchet Z_i 's,

$$X_t := \bigvee_{i=0}^{\infty} c_i Z_{t-i}, \quad t \in \mathbb{Z}$$

is well-defined stationary α -Fréchet process if $c_i \geq 0$ are such that

$$\sum_{i=0}^{\infty} c_i^\alpha < \infty.$$

- Observe that, $\forall x > 0$:

$$\mathbb{P}\{X_t \leq x\} = \prod_{i=0}^{\infty} \exp\{-(x/c_i)^{-\alpha}\} = \exp\{-\left(\sum_{i=0}^{\infty} c_i^\alpha\right) x^{-\alpha}\}.$$

- Similarly, one gets that

$$a_1 X_{t_1} \vee \cdots \vee a_d X_{t_d}$$

is α -Fréchet, for all $a_i \geq 0$, $t_i \in \mathbb{Z}$.

Fréchet and max-stable processes

The α -Fréchet processes are max-stable.

- How rich is the class of α -Fréchet processes?

Theorem (de Haan (1978))

A process X with α -Fréchet marginals is max-stable if and only if it is an α -Fréchet process.

- The class of Fréchet processes is **sufficient** for the study of ergodicity.
 - We focus on max-stable processes with Fréchet marginals or, equivalently, Fréchet processes.

The de Haan's spectral representation

Let $X = \{X_t\}_{t \in \mathbb{R}}$ be a continuous in probability α -Fréchet process.

- There exists a measure ρ on $(0, 1)$, and $f_t(u) \geq 0$, $\int_0^1 f_t^\alpha(u) \rho(du) < \infty$, such that

$$\{X_t\}_{t \in \mathbb{R}} \stackrel{fdd}{=} \left\{ \bigvee_{k=1}^{\infty} \frac{f_t(U_k)}{\epsilon_k^{1/\alpha}} \right\}_{t \in \mathbb{R}},$$

where $\{(U_k, \epsilon_k)\}$ is a Poisson point process on $(0, 1) \times (0, \infty)$ with intensity $\rho(du) \times dx$.

Note: Similar to the [LePage–Woodroffe–Zinn](#) series representation for α -stable processes (e.g. for $0 < \alpha < 1$):

$$\{Y_t\}_{t \in \mathbb{R}} \stackrel{d}{=} \left\{ \sum_{k=1}^{\infty} \frac{f_t(U_k)}{\epsilon_k^{1/\alpha}} \right\}_{t \in \mathbb{R}}$$

Fréchet random sup-measures

- Equivalently, one can use *extremal integrals*:

$$\{X_t\}_{t \in \mathbb{R}} \stackrel{fdd}{=} \left\{ \int_{(0,1)}^e f_t(u) M_\alpha(du) \right\}_{t \in \mathbb{R}}$$

where M_α is an α -Fréchet sup-measure on $(0, 1)$ with control measure $\rho(du)$.

Definition

M_α is α -Fréchet sup-measure on (E, \mathcal{E}) with control measure μ , if: (i) independently scattered (ii) $\mathbb{P}\{M_\alpha(A) \leq x\} = \exp\{-\mu(A)x^{-\alpha}\}$, $x > 0$ (iii) σ -sup-additive:

$$M_\alpha(\cup_{n=1}^{\infty} A_n) = \bigvee_{n=1}^{\infty} M_\alpha(A_n), \quad \text{almost surely.}$$

- Analogous to the α -stable measures in the sum-stable case (see Samorodnitsky and Taqqu (1994)).

Extremal integrals

Intuition: Let $A \cap B = \emptyset$ and $a, b > 0$. By independence, for $x > 0$

$$\begin{aligned}\mathbb{P}\{aM_\alpha(A) \vee bM_\alpha(B) \leq x\} &= e^{-\mu(A)(x/a)^{-\alpha}} e^{-\mu(B)(x/b)^{-\alpha}} \\ &= e^{-(a^\alpha \mu(A) + b^\alpha \mu(B))x^{-\alpha}}.\end{aligned}$$

For a simple function $f(u) = \sum_{k=1}^n a_k 1_{A_k}(u)$, $a_k \geq 0$, define

$$I(f) := \int_E^e f(u) M_\alpha(du) := \bigvee_{k=1}^n a_k M_\alpha(A_k),$$

to be the **extremal integral** of f w.r.t. M_α .

Extremal integrals: properties

- ① For simple $f(u) \geq 0$,

$$\mathbb{P}\{I(f) \leq x\} = e^{-\int_E f^\alpha d\mu \cdot x^{-\alpha}}, \quad x > 0.$$

- ② For constants $a, b > 0$, and simple $f(u), g(u) \geq 0$:

$$I(af \vee bg) = aI(f) \vee bI(g).$$

- ③ $I(f_i)$, $1 \leq i \leq d$ are independent **if and only if** f_i , $1 \leq i \leq d$ have disjoint supports (μ -a.e.).

Note: $\int_E f dM_\alpha$ can be defined for all measurable $f(u) \geq 0$, with $\int_E f^\alpha d\mu < \infty$.

Extremal integrals: benefits

- If $X_t = \vee_{i=0}^{\infty} c_i Z_{t-i}$, $t \in \mathbb{Z}$, then $X_t = \int_{\mathbb{R}}^e f(t-u) M_{\alpha}(du)$, with $M_{\alpha}(i-1, i] = Z_i$, $i \in \mathbb{Z}$, where $f(u) := \sum_{i=0}^{\infty} c_i 1_{[i, i+1)}(u)$, $0 \leq u < \infty$.
- General α -Fréchet processes:

$$X_t = \int_E^e f_t(u) M_{\alpha}(du), \quad t \in \mathbb{R}, \quad f_t(u) \in L_+^{\alpha}(E, \mathcal{E}, \mu(du)).$$

- Finite-dimensional distributions:

$$\begin{aligned} \mathbb{P}\{X_{t_j} \leq x_j, 1 \leq j \leq d\} &= \mathbb{P}\{\vee_{j=1}^d x_j^{-1} X_{t_j} \leq 1\} \\ \mathbb{P}\left\{\int_E^e (\vee_{j=1}^d x_j^{-1} f_{t_j}) d\mu \leq 1\right\} &= \exp\left\{-\int_E \left(\vee_{j=1}^d \frac{f_{t_j}}{x_j}\right)^{\alpha} d\mu\right\} \end{aligned}$$

- One can explicitly handle the fdd's via the *spectral functions* f_t !

Fréchet spaces and max-linear isometries

Consider $\mathcal{M} = \{I(f), f \in L_+^\alpha(\mu)\}$.

The class \mathcal{M} is closed w.r.t:

- max-linear combinations
- convergence in probability
- which is metrized by

$$\rho_{\mathcal{M}}(I(f), I(g)) := \rho_\alpha(f, g) := \int |f^\alpha - g^\alpha| d\mu$$

(see Davis and Resnick (1993) – context $\alpha = 1$).

Definition

$U : L_+^\alpha(\mu) \rightarrow L_+^\alpha(\mu)$ is a max-linear isometry if:

(i) $U(af \vee bg) = aU(f) \vee bU(g)$

(ii) $\int f^\alpha d\mu = \int U(f)^\alpha d\mu$.

Problem formulation

The process: Let $X = \{X_t\}_{t \in \mathbb{R}}$,

$$X_t := \int_E^e U_t(f) dM_\alpha, \quad -\infty < t < \infty,$$

where $f \in L_+^\alpha(\mu)$ and $\{U_t\}$ is a group of *max-linear isometries* on $L_+^\alpha(\mu)$.

• Then X is a stationary, α -Fréchet process.

Intuition: For $f(u) \in L_+^\alpha(du)$, think $U_t(f)(u) = f(t + u)$!

Goals: we look for *necessary and sufficient conditions* (on $\{U_t\}$ and f) for ergodicity/mixing of X .

Notes:

◦ The max-linear isometries $\{U_t\}$ are essentially the **pistons** of de Haan and Pickands (see S. (2007)).

◦ $X = \{X_t\}_{t \in \mathbb{R}}$ has a measurable modification if and only if it is continuous in probability (see S. (2007)).

General results: ergodicity

Theorem (S. (2007))

$X = \{X_t\}_{t \in \mathbb{R}}$ is ergodic, if and only if, for some (any) $p > 0$,

$$\frac{1}{T} \int_0^T \|U_\tau g \wedge g\|_{L^\alpha(\mu)}^p d\tau \longrightarrow 0, \quad \text{as } T \rightarrow \infty,$$

for all $g \in F := \overline{\vee - \text{span}\{U_t f, t \in \mathbb{R}\}}$.

- Here $F \subset L_+^\alpha(\mu)$ contains all positive max-linear combinations of $U_t f$'s and is closed in the metric ρ_α .
- Namely, all limits in ρ_α of the max-linear combinations

$$a_1 U_{t_1}(f) \vee \cdots \vee a_n U_{t_n}(f), \quad \forall a_i \geq 0, t_i \in \mathbb{R}.$$

General results: mixing

Theorem (S. (2007))

$X = \{X_t\}_{t \in \mathbb{R}}$ is mixing, if and only if,

$$\|U_\tau h \wedge g\|_{L_+^\alpha(\mu)} \longrightarrow 0, \quad \text{as } \tau \rightarrow \infty,$$

for all $g, h \in F = \overline{\text{span}}\{U_t f, t \in \mathbb{R}\}$.

- The proofs borrow ideas from the α –(sum)stable case – **Cambanis, Hardin and Weron (1987)**.

Key Idea: $\|f \wedge g\|_{L_+^\alpha(\mu)}$ 'measures' the dependence b/w
 $\xi = \int_E f dM_\alpha$ and $\eta = \int_E g dM_\alpha$.

Outline of the proof (mixing)

It is enough to focus on events of the type:

$$A = \{X_{s_i} \leq x_i, 1 \leq i \leq d\} \quad \text{and} \quad B = \{X_{t_i} \leq y_i, 1 \leq i \leq d\},$$

with $s_i, t_i \in \mathbb{R}$ and $x_i, y_i \geq 0$.

- Then, $B_\tau = \{X_{\tau+t_i} \leq y_i, 1 \leq i \leq d\}$ and also:

$$\mathbb{P}(A) = \mathbb{P}\{\bigvee_{i=1}^d x_i^{-1} X_{s_i} \leq 1\} = \exp\left\{-\int_E g^\alpha d\mu\right\}$$

$$\mathbb{P}(B) = \exp\left\{-\int_E h^\alpha d\mu\right\},$$

where

$$g(u) = \bigvee_{i=1}^d x_i^{-1} f_{s_i}(u) \quad \text{and} \quad h(u) = \bigvee_{i=1}^d y_i^{-1} f_{t_i}(u).$$

- Note that

$$\mathbb{P}(B_\tau) = \mathbb{P}\{\bigvee_{i=1}^d y_i^{-1} X_{\tau+t_i} \leq 1\} = \exp\left\{-\int_E U_\tau(h)^\alpha d\mu\right\}.$$

Proof ...

and, similarly, by the max-linearity of the extremal integrals:

$$\mathbb{P}(A \cap B_\tau) = \exp\left\{-\int_E (g^\alpha \vee U_\tau(h)^\alpha) d\mu\right\}.$$

- Thus $\mathbb{P}(A \cap B_\tau)/\mathbb{P}(A)\mathbb{P}(B) \rightarrow 1$ as $\tau \rightarrow \infty$ *if and only if*

$$\begin{aligned} & \exp\left\{\int_E g^\alpha d\mu + \int_E h^\alpha d\mu - \int_E g^\alpha \vee U_\tau(h)^\alpha d\mu\right\} \\ &= \exp\left\{\int_E g^\alpha \wedge U_\tau(h)^\alpha d\mu\right\} \longrightarrow 1 \quad (\tau \rightarrow \infty) \quad \square \end{aligned}$$

A 'natural' measure of dependence

Definition

For jointly α -Fréchet ξ and η ,

$$d_\alpha(\xi, \eta) := \|\xi\|_\alpha^\alpha + \|\eta\|_\alpha^\alpha - \|\xi \vee \eta\|_\alpha^\alpha,$$

is a measure of the dependence between ξ and η .

- Here $\|\xi\|_\alpha$ is the **scale** of ξ :

$$\mathbb{P}\{\xi \leq x\} = \exp\{-\|\xi\|_\alpha^\alpha x^{-\alpha}\}, \quad (x > 0).$$

- ξ and η are independent *if and only if* $d_\alpha(\xi, \eta) = 0$.
- If $\xi = \int_E f dM_\alpha$ and $\eta = \int_E g dM_\alpha$ then

$$d_\alpha(\xi, \eta) = \int_E f^\alpha d\mu + \int_E g^\alpha d\mu - \int_E f^\alpha \vee g^\alpha d\mu = \int_E f^\alpha \wedge g^\alpha d\mu$$

Mixing: a simpler criterion

Theorem (S. (2007))

A continuous in probability stationary α -Fréchet process $X = \{X_t\}_{t \in \mathbb{R}}$ is mixing if and only if $d_\alpha(X_t, X_0) \rightarrow 0$, as $t \rightarrow \infty$.

- The condition $d_\alpha(X_\tau, X_0) \rightarrow 0$, $\tau \rightarrow 0$ is **easy to check**.
- One and the same process $X = \{X_t\}$ may have **different representations** $(f, \{U_t\}, M_\alpha)$, but has **only one** dependence function

$$d_\alpha(\tau) = d_\alpha(X_\tau, X_0) = \|X_\tau\|_\alpha^\alpha + \|X_0\|_\alpha^\alpha - \|X_\tau \vee X_0\|_\alpha^\alpha.$$

Moving-maxima

- (*moving maxima*) Let $f \in L_+^\alpha(dx)$ and $\mu(dx) = dx$. Then, the process

$$X_t := \int_{\mathbb{R}} f(t+x) M_\alpha(dx), \quad -\infty < t < \infty,$$

is *mixing*. Indeed,

$$d_\alpha(\tau) = \int_{\mathbb{R}} f^\alpha(\tau+x) \wedge f^\alpha(x) dx \longrightarrow 0, \quad \text{as } \tau \rightarrow \infty.$$

- (*mixed moving maxima*) (M3 processes of Smith and Weissman) For $f(t, v) \in L_+^\alpha(dx \times \nu(dv))$ and $\mu(dt, dv) = dt \times \nu(dv)$, the process

$$X_t := \int_{\mathbb{R} \times V} f(t+x, v) M_\alpha(dx, dv), \quad -\infty < t < \infty,$$

is *mixing*.

Non-ergodic processes

- (*non-ergodicity*) Let (E, \mathcal{E}, μ) be $([0, 2\pi), \mathcal{B}, dx)$ and

$$X_t = \int_{[0, 2\pi)}^e \sin^2(t + x) M_\alpha(dx), \quad -\infty < t < \infty.$$

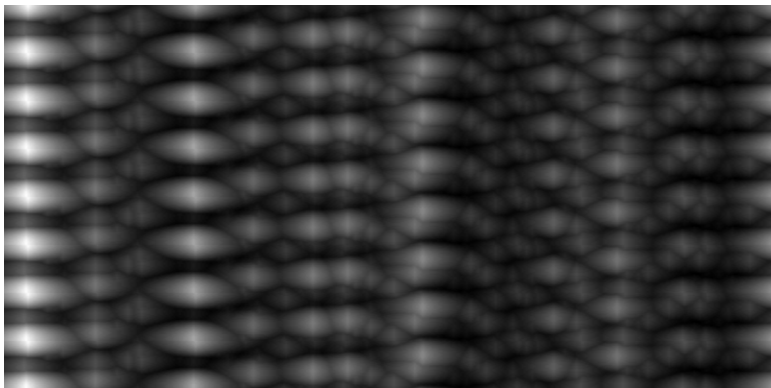
The process $X = \{X_t\}_{t \in \mathbb{R}}$ is strictly stationary and **non-ergodic**.

- (*a field*) The stationary and non-isotropic field

$$X(s, t) = \int_{[0, \pi) \times \mathbb{R}}^e \sin^2(s + u) \times e^{-|t+v|} M_\alpha(du, dv)$$

is **non-ergodic** along “s” but **ergodic** along “t”.

An illustration



- A movie ...

An example of Brown and Resnick

Let (E, \mathcal{E}, μ) be a *probability space* and $w = \{w_t\}_{t \geq 0}$ be a standard Brownian motion there.

- Define the *doubly stochastic process*

$$X_t = \int_E e^{w_t(u) - \alpha t/2} M_\alpha(du), \quad 0 \leq t < \infty,$$

where M_α , $\alpha > 0$ is defined on a different probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- Brown and Resnick (1977) introduced $X = \{X_t\}_{t \geq 0}$ and showed that it is stationary.

Theorem (S. (2007))

X is exponentially mixing.

- Namely, $d_\alpha(\tau) \leq e^{-c\tau}$, $\tau > 0$, for some $c > 0$.

Estimation of the dependence function

Let $\{X_t\}_{t \in \mathbb{R}}$ be **stationary** and **ergodic**.

Problem: Estimate the dependence function

$$d_\alpha(\tau) := d_\alpha(X_\tau, X_0).$$

- A consistent estimator of the **dependence function** is:

$$\widehat{d}_{\alpha,p,n}(\tau) := c_{p,\alpha} \left(\frac{2}{n} \sum_{k=1}^n X_k^p \right)^{\alpha/p} - c_{p,\alpha} \left(\frac{1}{n-\tau} \sum_{k=1}^{n-\tau} (X_{\tau+k} \vee X_k)^p \right)^{\alpha/p},$$

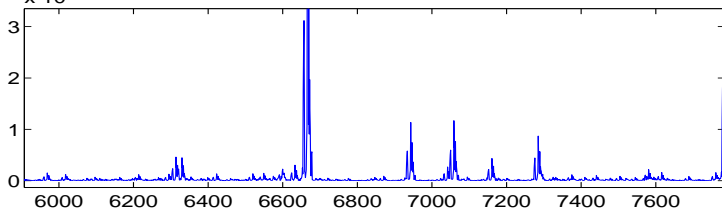
where $0 < p < \alpha$ and $c_{p,\alpha} = \Gamma(1 - p/\alpha)^{-\alpha/p}$.

- By ergodicity, for all $\gamma \in (0, 1)$

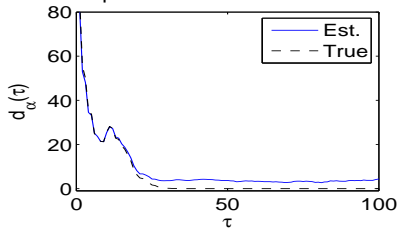
$$\widehat{d}_{\alpha,p,n}(\tau) \xrightarrow{a.s.} d_\alpha(\tau) \quad \text{and} \quad \mathbb{E} |\widehat{d}_{\alpha,p,n}(\tau) - d_\alpha(\tau)|^\gamma \rightarrow 0, \quad \text{as } n \rightarrow \infty.$$

A numerical example

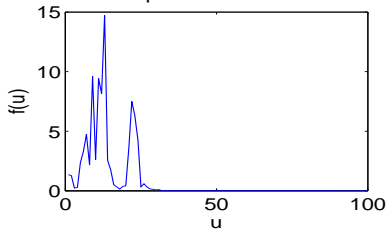
$\times 10^4$ Moving Maxima Time Series (Part): $\alpha = 1$, $N=2^{15} = 32,768$



Dependence Function Estimate



Spectral Function



Concluding remarks

- Weintraub (1991) introduced 3 notions of “mixing” in terms of the de Haan’s spectral representation.
 - No connections with the classical notions of ergodicity and mixing was established.
- We show that Weintraub’s ‘0–mixing’ is equivalent to mixing.
- Our work justifies and suggests a range of statistical methods for max–stable processes.
- Some old new tools on modeling and statistics for max–stable processes/fields?
- Further questions on:
 - estimation of the spectral function
 - representations of max–stable processes
 - random fields

Some references

Brown, B. M. & Resnick, S. I. (1977), 'Extreme values of independent stochastic processes', *J. Appl. Probability* **14**(4), 732–739.

Cambanis, S., Hardin, Jr., C. D. & Weron, A. (1987), 'Ergodic properties of stationary stable processes', *Stoch. Proc. Appl.* **24**, 1–18.

Davis, R. & Resnick, S.I. (1993), 'Prediction of stationary max–stable processes', *Ann. Appl. Probab.*, **3**(2), 497–525.

de Haan, L. (1978), 'A characterization of multidimensional extreme-value distributions', *Sankhyā (Statistics). The Indian Journal of Statistics. Series A* **40**(1), 85–88.

de Haan, L. (1984), 'A spectral representation for max–stable processes', *Ann. Probab.* **12**(4), 1194–1204.

Stoev, S. & Taqqu, M. S. (2005), 'Extremal stochastic integrals: a parallel between max–stable processes and α –stable processes', *Extremes* **8**, 237–266.

Stoev, S. (2007), 'On the ergodicity and mixing of max–stable processes', *Stoch. Proc. Appl.* To appear.

Weintraub, K. S. (1991), 'Sample and ergodic properties of some min–stable processes', *Ann. Probab.* **19**(2), 706–723.