On the estimation of the heavy–tail exponent in time series using the max–spectrum

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Outline

- Heavy–tailed time series
- Scaling of maxima: Tail exponent α and extremal index θ
- The max–spectrum: Estimation of α and θ
- Some asymptotic results
- Applications

Preliminaries

A r.v. X has a heavy right tail with tail exponent $\alpha > 0$, if:

$$\mathbb{P}\{X > x\} \sim L(x)x^{-\alpha}, \quad \text{as } x \to \infty,$$

where $L(x)$ is a slowly varying function.

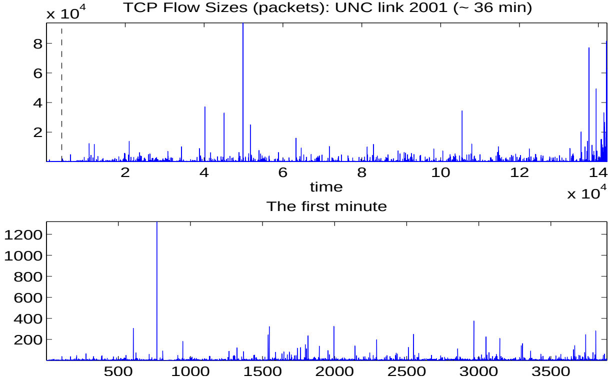
- For simplicity, here we focus on positive X with $L(x) \equiv \mathrm{const}$.
- Thus, for $p > 0$, the moment:

$$\mathbb{E}X^p < \infty \quad \text{if and only if} \quad p < \alpha.$$

- Heavy tailed models are natural in many applications: physics, meteorology, insurance, finance, telecommunications, etc.
- Pareto, $\alpha-$stable, $t-$distributions, Cauchy are all heavy–tailed.

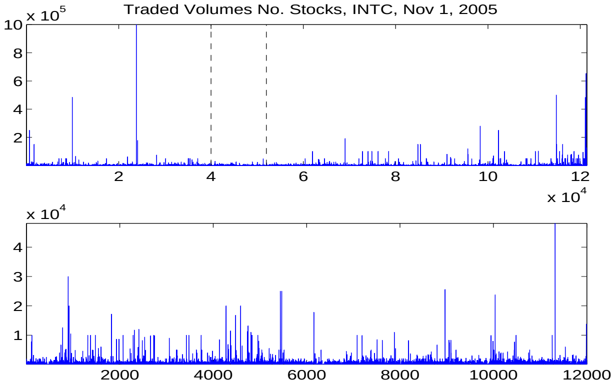
Network traffic data

- Transmission Control Protocol (TCP) connection sizes (in # packets)



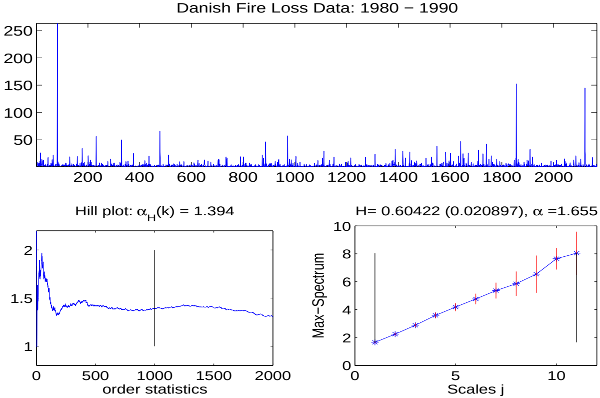
Traded volume data

- Traded volume per individual transactions (in # shares)



Insurance data

- Insurance claims due to fire–loss (in MLN Danish Krones)



Scaling of maxima: Independent data

Consider iid, positive, and heavy–tailed $X_1, X_2, \ldots, X_n, \ldots$. Let

$$M_n = \max_{1 \le i \le n} X_i \equiv \bigvee_{1 \le i \le n} X_i.$$

If $\mathbb{P}\{X > x\} \sim cx^{-\alpha}, \ x \to \infty$, then

$$\frac{1}{n^{1/\alpha}} M_n \xrightarrow{d} Z, \quad \text{as } n \to \infty,$$

where Z has the $\alpha-$Fréchet distribution:

$$\mathbb{P}\{Z \le x\} = e^{-cx^{-\alpha}}, \quad (x > 0).$$

- See e.g. Resnick (1987).

Scaling of maxima: Dependent data

Let now $\{X_k\}_{k\in\mathbb{Z}}$ be a stationary, heavy–tailed time series with tail exponent $\alpha > 0$ and

$$M_n = \max_{1\le i\le n} X_i \equiv \bigvee_{1\le i\le n} X_i.$$

How do the maxima M_n scale?

The seminal results of Leadbetter (see e.g. Leadbetter, Lindgren & Rootzén (1983)) imply that under mild dependence conditions "$D(u_n)$":

$$\frac{1}{n^{1/\alpha}} M_n \stackrel{d}{\longrightarrow} \theta^{1/\alpha} Z, \quad \text{with } \theta \in [0,1],$$

where for iid $\{X_k^*\}_{k\in\mathbb{Z}}$, with $X_i =^d X_i^*$, we have

$$\frac{1}{n^{1/\alpha}} \bigvee_{1\le i\le n} X_i^* \stackrel{d}{\longrightarrow} Z, \quad \text{as } n\to\infty.$$

- Here $\theta \in [0,1]$ is called the extremal index of the time series $\{X_k\}_{k\in\mathbb{Z}}$.

More on the extremal index θ

- We always have $0 \le \theta \le 1$.
- If $\theta > 0$, then the maxima M_n scale at the same rate "$n^{1/\alpha}$" for dependent and independent data.
- For independent data $\theta = 1$, but not conversely.
- The extremal index is related to the clustering of extremes phenomenon.
- Under general conditions, the exceedence times of $\{X_k\}_{k\in\mathbb{Z}}$ converge to a cluster Poisson point process:

$$\theta = \frac{1}{\text{(expected cluster size)}}.$$

- The estimation of α and θ is important in many applications.

Max–spectrum: Definition

Data: $X_i,\ 1 \le i \le n$ (Think: stationary time series $\{X_k\}_{k\in\mathbb{Z}}$)
Dyadic Block maxima: Define

$$C_{j,k} = \bigvee_{1\le i\le 2^j} X_{2^j(k-1)+i}.$$

- Illustration:

$$\underbrace{\underbrace{X_1,\ X_2}_{C_{1,1},},\ \underbrace{X_3,\ X_4}_{C_{1,2},}}_{C_{2,1}},\cdots$$

- Introduce the statistics:

$$Y_j := \frac{1}{n_j}\sum_{k=1}^{n_j} \log_2 C_{j,k}, \quad \text{where } n_j = [n/2^j].$$

Max–spectrum: properties

- the set of statistics:

$$Y_j := \frac{1}{n_j} \sum_{k=1}^{n_j} \log_2 C_{j,k}, \quad \text{where } n_j = [n/2^j], \quad \text{and } 1 \le j \le \log_2 n$$

is said to be the max–spectrum of the data $X_1, \dots, X_n$.
The scaling of maxima implies:

$$Y_j \overset{P}{\sim} \frac{1}{\alpha} j + \frac{1}{\alpha} \log_2 \theta + \mathbf{const}, \tag{1}$$

as both j and $n \to \infty$.

- the max–spectrum is asymptotically linear even for dependent data, as long as the extremal index θ is positive.
- The tail index α can be readily estimated from the Y_j's.
- With more work, one can also get θ. Note that the **const** in (1) is unknown.

Estimating α

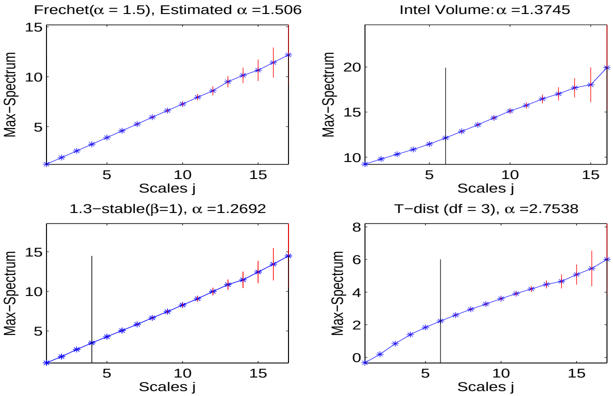
Consider the max–spectrum $Y_j,\ 1 \le j \le J$ of a heavy–tailed time series and define:

$$1/\widehat{\alpha}(j_1, j_2) := \sum_{j_1 \le j \le j_2} w_j Y_j,$$

where $\sum_{j_1 \le j \le j_2} w_j = 0$ and $\sum_{j_1 \le j \le j_2} j w_j = 1$.

- $1/\alpha$ is the slope of the max-spectrum:

Max–spectra: an illustration



Estimating θ

For non–iid time series data, we have:

$$Y_j \overset{P}{\sim} \frac{1}{\alpha} j + \frac{1}{\alpha} \log_2 \theta + \mathbf{const}, \quad \text{as } j \text{ and } n \to \infty. \tag{2}$$

How to get rid of the **const**?

- Use resampling, i.e. consider

$$X_1^*, X_2^*, \cdots, X_n^*, \quad \text{which is}$$

either a random permutation or a bootstrap sample of the data. For iid X_i^*'s with the same marginal as X_i's, the max–spectrum is:

$$Y_j^* \overset{P}{\sim} \frac{1}{\alpha} j + \frac{1}{\alpha} + \mathbf{const}, \quad \text{as } j \text{ and } n \to \infty. \tag{3}$$

- The **const** in (2) and (3) are equal and θ in (3) is gone!

Estimating θ (cont'd)

On scale j, the point estimate of θ is:

$$\widehat{\theta}(j) = 2^{\widehat{\alpha}\times(Y_j^* - Y_j)}, \quad \text{for some } \widehat{\alpha}.$$

• But one can also resample multiple times to get a sample $\widehat{\theta}_i(j),\ \ i = 1, \ldots, m.$

What scale j to choose? Ans: $j = j(n) \to \infty$ should grow with n.

More difficult questions:

◦ What is the optimal rate of $j(n)$?

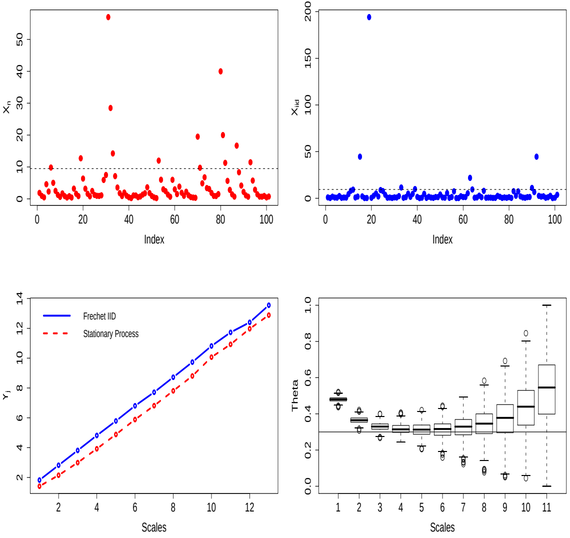
◦ What is the role of $m = m(n)$?

◦ What is the asymptotic distribution of $\widehat{\alpha}(j_1, j_2)$?

◦ What is the asymptotic distribution of $\widehat{\theta}(j)$?

Some answers to come...

Estimating θ: an illustration



Asymptotic results: for iid X_i's

One can show the asymptotic normality for $\widehat{\alpha}(j_1, j_2)$.
The key condition is on the second order asymptotics of the tails:

$$|x^{\alpha}\mathbb{P}\{X_i > x\} - C| \le Dx^{-\beta}, \quad \text{for large } x, \tag{4}$$

and some $\beta > 0$.
The asymptotic regime:

- Let $(j_1, j_2) = (1, k) + r(n)$, for fixed k.
- Consider the max–spectrum vector

$$\vec{Y}_r = (Y_{r+j})_{j=1}^{k}, \quad \text{for the range of scales } (j_1(n), j_2(n)).$$

- The next result is a uniform CLT for $\vec{Y}_r$.

CLT for the max–spectrum: iid case

Theorem(S., Michailidis & Taqqu) *Under (4) and another mild condition, for any* $\vec{v} \in \mathbb{R}^k$*:*

$$\sup_{x\in\mathbb{R}} \Big|\mathbb{P}\{\sqrt{n/2^r}\Big((\vec{v},\vec{Y}_r) - (\vec{v},\vec{\mu}_r)\Big) \le x\} - \Phi(x/\sigma_{\vec{v}})\Big| \le C_{\vec{v}}\Big(1/2^{r\beta/\alpha} + r2^{r/2}/\sqrt{n}\Big), \quad (5)$$

where Φ *is the standard Normal c.d.f.*

- For the mean $\vec{\mu}_r = (\mu_r(j))_{j=1}^k$ and the variance, we have:

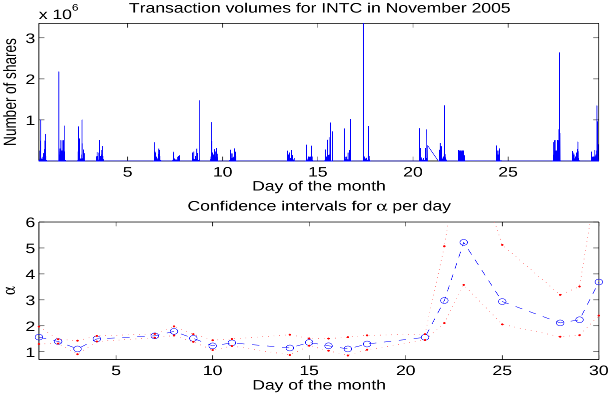
$$\mu_r(j) = (r+j)/\alpha + \mathbf{const}, \quad \text{and} \quad \sigma_{\vec{v}}^2 = (\vec{v}, \Sigma_\alpha \vec{v}).$$

- The asymptotic covariance matrix Σ_α can be identified!
- The proof involves Berry–Esseen and sharp rates for $\mathbb{E}f(n^{-1/\alpha}\max_{1\le i\le n} X_i)$ with $f(\cdot) = \log_2(\cdot)$.
- The uniform CLT is more informative than the usual CLT!

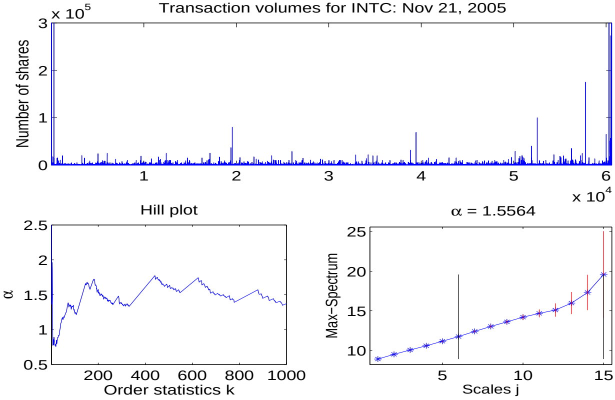
Dependent data

- We have asymptotic normality results for $\widehat{\theta}$ and $\widehat{\alpha}$ for $m-$dependent time series.
- The general case is under investigation.
- We have results on:
  - confidence intervals
  - automatic selection of scales (j_1, j_2).
  - diagnostics

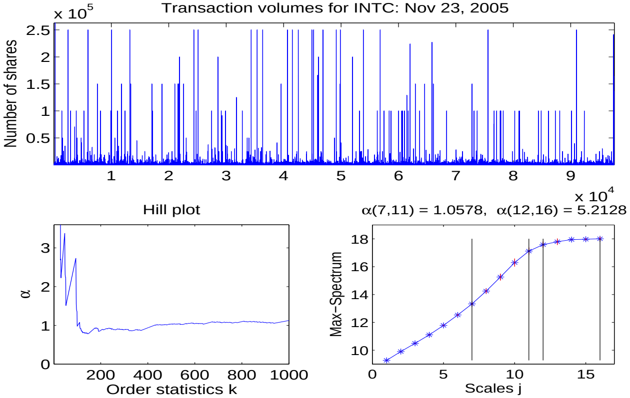
Intel Stock: traded volumes November 2005



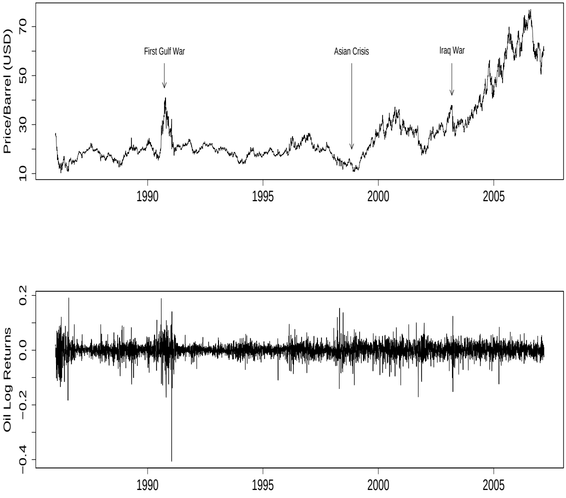
Intel Stock: traded volumes, November 21, 2005



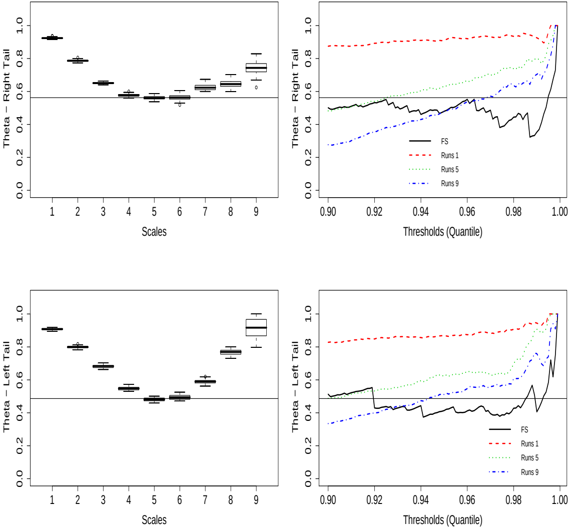
Intel Stock: traded volumes, November 23, 2005



Oil data


Price/Barrel (USD)
10
30
50
70
First Gulf War
Asian Crisis
Iraq War
1990
1995
2000
2005
Oil Log Returns
−0.4
−0.2
0.0
0.2
1990
1995
2000
2005

Oil returns: extremal index estimates



References

Leadbetter, M. R., Lindgren, G., and Rootzén, H. (1983). *Extremes and Related Properties of Random Sequences and Processes*. Springer-Verlag, New York.

Resnick, S. I. (1987). *Extreme Values, Regular Variation and Point Processes*. Springer-Verlag, New York.