Max–stable processes: representations, ergodic properties and some statistical applications

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The focus

Let $X = \{X_t\}_{t\in\mathbb{R}}$ be a (strictly) stationary max–stable process.

Questions:

- When is X ergodic or mixing?
- How can one check ergodicity/mixing for X?

Answers: Must be in terms of representations for max–stable processes.

Max–stable processes

The process $X = \{X_t\}_{t\in\mathbb{R}}$ is a max–stable process if its finite–dimensional distributions (fdd) are multivariate max–stable:

Definition

A vector $\mathbf{X} = \{X_j\}_{1\le j\le d}$ is max–stable if for all $k \in \mathbb{N}$, there exist $\mathbf{a}_k \in \mathbb{R}^d_+$, $\mathbf{b}_k \in \mathbb{R}^d$, such that

$$\frac{1}{\mathbf{a}_k}\bigvee_{i=1}^{k}\mathbf{X}^{(i)} - \mathbf{b}_k \stackrel{d}{=} \mathbf{X},$$

where $\mathbf{X}^{(i)}$, $1 \le i \le k$ are independent copies of $\mathbf{X}$.

- The marginals of X are either *Fréchet*, *Gumbel* or *negative Fréchet*.

Fréchet processes

- ξ is an $\alpha-$Fréchet variable ($\alpha > 0$) with scale $\sigma > 0$, if

$$\mathbb{P}\{\xi \le x\} = \exp\{-(x/\sigma)^{-\alpha}\} = \exp\{-\sigma^\alpha x^{-\alpha}\}, \quad x \in (0, \infty).$$

◦ $\mathbb{P}\{\xi > x\} \sim \sigma^\alpha x^{-\alpha}$, as $x \to \infty$.

Definition

$\{X_t\}_{t\in\mathbb{R}}$ is an $\alpha-$*Fréchet process* if for all $a_i > 0$ and t_i, $1 \le i \le d$, the *max–linear combinations*

$$\max\{a_i X_{t_i},\ 1 \le i \le d\} = \bigvee_{i=1}^{d} a_i X_{t_i},$$

are $\alpha-$Fréchet.

Examples

- (moving maxima) For iid $\alpha-$Fréchet Z_i's,

$$X_t := \bigvee_{i=0}^{\infty} c_i Z_{t-i}, \quad t \in \mathbb{Z}$$

is well–defined stationary $\alpha-$Fréchet process if $c_i \geq 0$ are such that

$$\sum_{i=0}^{\infty} c_i^{\alpha} < \infty.$$

- Observe that, $\forall x > 0$:

$$\mathbb{P}\{X_t \leq x\} = \prod_{i=0}^{\infty} \exp\{-(x/c_i)^{-\alpha}\} = \exp\{-(\sum_{i=0}^{\infty} c_i^{\alpha}) x^{-\alpha}\}.$$

- Similarly, one gets that

$$a_1 X_{t_1} \vee \cdots \vee a_d X_{t_d}$$

is $\alpha-$Fréchet, for all $a_i \geq 0$, $t_i \in \mathbb{Z}$.

Fréchet and max–stable processes

The $\alpha-$Fréchet processes are max–stable.

• How rich is the class of $\alpha-$Fréchet processes?

Theorem (de Haan (1978))

A process X with $\alpha-$Fréchet marginals is max–stable if and only if it is an $\alpha-$Fréchet process.

• The class of Fréchet processes is sufficient for the study of ergodicity.

◦ We focus on max-stable processes with Fréchet marginals or, equivalently, Fréchet processes.

The de Haan's spectral representation

Let $X = \{X_t\}_{t\in\mathbb{R}}$ be a continuous in probability $\alpha-$Fréchet process.

- There exists a measure ρ on $(0,1)$, and $f_t(u) \geq 0,\ \int_0^1 f_t^\alpha(u)\rho(du) < \infty$, such that

$$\{X_t\}_{t\in\mathbb{R}} \stackrel{fdd}{=} \left\{ \bigvee_{k=1}^{\infty} \frac{f_t(U_k)}{\epsilon_k^{1/\alpha}} \right\}_{t\in\mathbb{R}},$$

where $\{(U_k, \epsilon_k)\}$ is a Poisson point process on $(0,1)\times(0,\infty)$ with intensity $\rho(du)\times dx$.

Note: Similar to the LePage–Woodroofe–Zinn series representation for $\alpha-$stable processes (e.g. for $0 < \alpha < 1$):

$$\{Y_t\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{ \sum_{k=1}^{\infty} \frac{f_t(U_k)}{\epsilon_k^{1/\alpha}} \right\}_{t\in\mathbb{R}}$$

Fréchet random sup–measures

- Equivalently, one can use *extremal integrals*:

$$\{X_t\}_{t\in\mathbb{R}} \stackrel{fdd}{=} \Big\{ \int^{e}_{(0,1)} f_t(u) M_\alpha(du) \Big\}_{t\in\mathbb{R}}$$

where M_α is an $\alpha-$Fréchet sup–measure on $(0,1)$ with control measure $\rho(du)$.

Definition

M_α is $\alpha-$Fréchet sup–measure on $(E,\mathcal{E})$ with control measure μ, if: *(i)* independently scattered *(ii)* $\mathbb{P}\{M_\alpha(A) \le x\} = \exp\{-\mu(A)x^{-\alpha}\},\ x > 0$ *(iii)* $\sigma-$sup–additive:

$$M_\alpha(\cup_{n=1}^{\infty} A_n) = \bigvee_{n=1}^{\infty} M_\alpha(A_n), \quad \text{almost surely.}$$

- Analogous to the $\alpha-$stable measures in the sum–stable case (see Samorodnitsky and Taqqu (1994)).

Extremal integrals

Intuition: Let $A \cap B = \emptyset$ and $a, b > 0$. By independence, for $x > 0$

$$\begin{aligned} \mathbb{P}\{aM_\alpha(A) \vee bM_\alpha(B) \le x\} &= e^{-\mu(A)(x/a)^{-\alpha}} e^{-\mu(B)(x/b)^{-\alpha}} \\ &= e^{-(a^\alpha \mu(A) + b^\alpha \mu(B)) x^{-\alpha}}. \end{aligned}$$

For a simple function $f(u) = \sum_{k=1}^n a_k 1_{A_k}(u)$, $a_k \ge 0$, define

$$I(f) := {\int_E}^{e} f(u) M_\alpha(du) := \bigvee_{k=1}^n a_k M_\alpha(A_k),$$

to be the extremal integral of f w.r.t. M_α.

Extremal integrals: properties

1. For simple $f(u) \geq 0$,

$$\mathbb{P}\{I(f) \leq x\} = e^{-\int_E f^\alpha d\mu \cdot x^{-\alpha}}, \; x > 0.$$

2. For constants $a, b > 0$, and simple $f(u), g(u) \geq 0$:

$$I(af \vee bg) = aI(f) \vee bI(g).$$

3. $I(f_i), \; 1 \leq i \leq d$ are independent if and only if $f_i, 1 \leq i \leq d$ have disjoint supports (μ−a.e.).

Note: ${}^{e}\!\!\int_E f dM_\alpha$ can be defined for all measurable $f(u) \geq 0$, with $\int_E f^\alpha d\mu < \infty$.

Extremal integrals: benefits

- If $X_t = \vee_{i=0}^{\infty} c_i Z_{t-i},\ t \in \mathbb{Z}$, then $X_t = {}^{e}\!\!\int_{\mathbb{R}} f(t-u) M_\alpha(du)$, with $M_\alpha(i-1, i] = Z_i,\ i \in \mathbb{Z}$, where $f(u) := \sum_{i=0}^{\infty} c_i 1_{[i,i+1)}(u)$, $0 \le u < \infty$.
- General $\alpha-$Fréchet processes:

$$X_t = {}^{e}\!\!\int_E f_t(u) M_\alpha(du), \quad t \in \mathbb{R},\ f_t(u) \in L_+^\alpha(E, \mathcal{E}, \mu(du)).$$

- Finite–dimensional distributions:

$$\begin{aligned} \mathbb{P}\{X_{t_j} \le x_j,\ 1 \le j \le d\} &= \mathbb{P}\{\vee_{j=1}^d x_j^{-1} X_{t_j} \le 1\} \\ \mathbb{P}\{ {}^{e}\!\!\int_E (\vee_{j=1}^d x_j^{-1} f_{t_j}) d\mu \le 1\} &= \exp\Big\{ - \int_E \Big(\bigvee_{j=1}^d \frac{f_{t_j}}{x_j} \Big)^\alpha d\mu \Big\} \end{aligned}$$

- One can explicitly handle the fdd's via the *spectral functions* f_t!

Fréchet spaces and max–linear isometries

Consider $\mathcal{M} = \{I(f),\ f \in L^{\alpha}_{+}(\mu)\}$.
The class $\mathcal{M}$ is closed w.r.t:

- max–linear combinations
- convergence in probability

○ which is metrized by

$$\rho_{\mathcal{M}}(I(f), I(g)) := \rho_{\alpha}(f, g) := \int |f^{\alpha} - g^{\alpha}| d\mu$$

(see Davis and Resnick (1993) – context $\alpha = 1$).

Definition

$U : L^{\alpha}_{+}(\mu) \longrightarrow L^{\alpha}_{+}(\mu)$ is a max–linear isometry if:
(i) $U(af \vee bg) = aU(f) \vee bU(g)$
(ii) $\int f^{\alpha} d\mu = \int U(f)^{\alpha} d\mu$.

Problem formulation

The process: Let $X = \{X_t\}_{t\in\mathbb{R}}$,

$$X_t := \int_E^e U_t(f) dM_\alpha, \quad -\infty < t < \infty,$$

where $f \in L_+^\alpha(\mu)$ and $\{U_t\}$ is a group of *max–linear isometries* on $L_+^\alpha(\mu)$.
• Then X is a stationary, $\alpha-$Fréchet process.
Intuition: For $f(u) \in L_+^\alpha(du)$, think $U_t(f)(u) = f(t+u)$!
Goals: we look for *necessary and sufficient conditions* (on $\{U_t\}$ and f) for ergodicity/mixing of X.
Notes:
◦ The max–linear isometires $\{U_t\}$ are essentially the pistons of de Haan and Pickands (see S. (2007)).
◦ $X = \{X_t\}_{t\in\mathbb{R}}$ has a measurable modification if and only if it is continuous in probability (see S. (2007)).

General results: ergodicity

Theorem (S. (2007))

$X = \{X_t\}_{t\in\mathbb{R}}$ is ergodic, if and only if, for some (any) $p > 0$,

$$\frac{1}{T}\int_0^T \|U_\tau g \wedge g\|_{L^\alpha(\mu)}^p d\tau \longrightarrow 0, \quad as\ T \to \infty,$$

for all $g \in F := \overline{\vee - span}\{U_t f,\ t \in \mathbb{R}\}$.

• Here $F \subset L_+^\alpha(\mu)$ contains all positive max–linear combinations of $U_t f$'s and is closed in the metric ρ_α.
◦ Namely, all limits in ρ_α of the max–linear combinations

$$a_1 U_{t_1}(f) \vee \cdots \vee a_n U_{t_n}(f), \quad \forall a_i \geq 0,\ t_i \in \mathbb{R}.$$

General results: mixing

Theorem (S. (2007))

$X = \{X_t\}_{t\in\mathbb{R}}$ is mixing, if and only if,

$$\|U_\tau h \wedge g\|_{L_+^\alpha(\mu)} \longrightarrow 0, \quad \text{as } \tau \to \infty,$$

for all $g, h \in F = \overline{\vee - span}\{U_t f,\ t \in \mathbb{R}\}$.

• The proofs borrow ideas from the $\alpha-$(sum)stable case – Cambanis, Hardin and Weron (1987).

Key Idea: $\|f \wedge g\|_{L_+^\alpha(\mu)}$ 'measures' the *dependence* b/w $\xi = {}^{e}\!\!\int_E f dM_\alpha$ and $\eta = {}^{e}\!\!\int_E g dM_\alpha$.

Outline of the proof (mixing)

It is enough to focus on events of the type:

$$A = \{X_{s_i} \le x_i,\ 1 \le i \le d\} \quad \text{and} \quad B = \{X_{t_i} \le y_i,\ 1 \le i \le d\},$$

with $s_i, t_i \in \mathbb{R}$ and $x_i, y_i \ge 0$.

- Then, $B_\tau = \{X_{\tau+t_i} \le y_i, 1 \le i \le d\}$ and also:

$$\begin{aligned}\mathbb{P}(A) &= \mathbb{P}\{\vee_{i=1}^d x_i^{-1} X_{s_i} \le 1\} = \exp\{-\int_E g^\alpha d\mu\} \\ \mathbb{P}(B) &= \exp\{-\int_E h^\alpha d\mu\},\end{aligned}$$

where

$$g(u) = \vee_{i=1}^d x_i^{-1} f_{s_i}(u) \quad \text{and} \quad h(u) = \vee_{i=1}^d y_i^{-1} f_{t_i}(u).$$

- Note that

$$\mathbb{P}(B_\tau) = \mathbb{P}\{\vee_{i=1}^d y_i^{-1} X_{\tau+t_i} \le 1\} = \exp\{-\int_E U_\tau(h)^\alpha d\mu\}.$$

Proof ...

and, similarly, by the max–linearity of the extremal integrals:

$$\mathbb{P}(A \cap B_\tau) = \exp\{-\int_E (g^\alpha \vee U_\tau(h)^\alpha) d\mu\}.$$

- Thus $\mathbb{P}(A \cap B_\tau)/\mathbb{P}(A)\mathbb{P}(B) \to 1$ as $\tau \to \infty$ *if and only if*

$$\exp\{\int_E g^\alpha d\mu + \int_E h^\alpha d\mu - \int_E g^\alpha \vee U_\tau(h)^\alpha d\mu\}$$
$$= \exp\{\int_E g^\alpha \wedge U_\tau(h)^\alpha d\mu\} \longrightarrow 1 \quad (\tau \to \infty) \; \square$$

A 'natural' measure of dependence

Definition

For jointly $\alpha-$Fréchet ξ and η,

$$d_\alpha(\xi,\eta) := \|\xi\|_\alpha^\alpha + \|\eta\|_\alpha^\alpha - \|\xi \vee \eta\|_\alpha^\alpha,$$

is a measure of the dependence between ξ and η.

- Here $\|\xi\|_\alpha$ is the scale of ξ:

$$\mathbb{P}\{\xi \le x\} = \exp\{-\|\xi\|_\alpha^\alpha x^{-\alpha}\}, \quad (x > 0).$$

- ξ and η are independent *if and only if* $d_\alpha(\xi,\eta) = 0$.
- If $\xi = {}^e\!\!\int_E f dM_\alpha$ and $\eta = {}^e\!\!\int_E g dM_\alpha$ then

$$d_\alpha(\xi,\eta) = \int_E f^\alpha d\mu + \int_E g^\alpha d\mu - \int_E f^\alpha \vee g^\alpha d\mu = \int_E f^\alpha \wedge g^\alpha d\mu$$

Mixing: a simpler criterion

Theorem (S. (2007))

A continuous in probability stationary $\alpha-$Fréchet process $X = \{X_t\}_{t\in\mathbb{R}}$ is mixing if and only if $d_\alpha(X_t, X_0) \to 0$, as $t \to \infty$.

- The condition $d_\alpha(X_\tau, X_0) \to 0,\ \tau \to 0$ is easy to check.
- One and the same process $X = \{X_t\}$ may have different representations $(f, \{U_t\}, M_\alpha)$, but has only one dependence function

$$d_\alpha(\tau) = d_\alpha(X_\tau, X_0) = \|X_\tau\|_\alpha^\alpha + \|X_0\|_\alpha^\alpha - \|X_\tau \vee X_0\|_\alpha^\alpha.$$

Moving–maxima

- *(moving maxima)* Let $f \in L_+^\alpha(dx)$ and $\mu(dx) = dx$. Then, the process

$$X_t := \int_{\mathbb{R}}^{e} f(t+x) M_\alpha(dx), \quad -\infty < t < \infty,$$

is mixing. Indeed,

$$d_\alpha(\tau) = \int_{\mathbb{R}} f^\alpha(\tau + x) \wedge f^\alpha(x) dx \longrightarrow 0, \quad \text{as } \tau \to \infty.$$

- *(mixed moving maxima)* (M3 processes of Smith and Weissman) For $f(t, v) \in L_+^\alpha(dx \times \nu(dv))$ and $\mu(dt, dv) = dt \times \nu(dv)$, the process

$$X_t := \int_{\mathbb{R} \times V}^{e} f(t + x, v) M_\alpha(dx, dv), \quad -\infty < t < \infty,$$

is mixing.

Non–ergodic processes

- *(non–ergodicity)* Let $(E, \mathcal{E}, \mu)$ be $([0, 2\pi), \mathcal{B}, dx)$ and

$$X_t = \int_{[0,2\pi)}^{e} \sin^2(t + x) M_\alpha(dx), \quad -\infty < t < \infty.$$

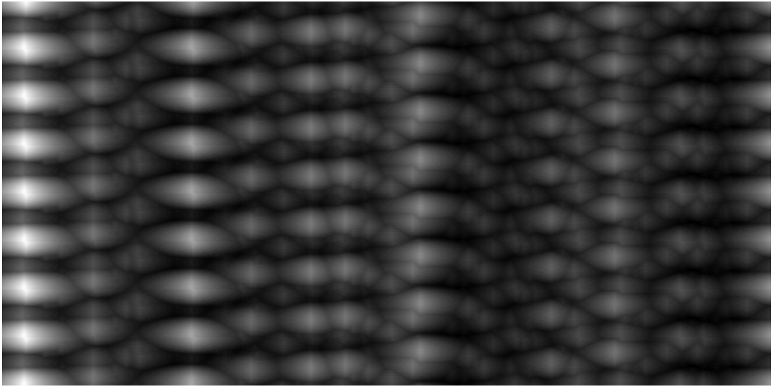
The process $X = \{X_t\}_{t \in \mathbb{R}}$ is strictly stationary and non–ergodic.

- *(a field)* The stationary and non–isotropic field

$$X(s, t) = \int_{[0,\pi)\times\mathbb{R}}^{e} \sin^2(s + u) \times e^{-|t+v|} M_\alpha(du, dv)$$

is non–ergodic along “s” but ergodic along “t”.

An illustration



- A movie ...

An example of Brown and Resnick

Let $(E, \mathcal{E}, \mu)$ be a *probability space* and $w = \{w_t\}_{t\geq 0}$ be a standard Brownian motion there.

- Define the *doubly stochastic process*

$$X_t = \int_E^e e^{w_t(u) - \alpha t/2} M_\alpha(du), \quad 0 \leq t < \infty,$$

where M_α, $\alpha > 0$ is defined on a different probability space $(\Omega, \mathcal{F}, \mathbb{P})$.

- Brown and Resnick (1977) introduced $X = \{X_t\}_{t\geq 0}$ and showed that it is stationary.

Theorem (S. (2007))

X is exponentially mixing.

- Namely, $d_\alpha(\tau) \leq e^{-c\tau}$, $\tau > 0$, for some $c > 0$.

Estimation of the dependence function

Let $\{X_t\}_{t\in\mathbb{R}}$ be stationary and ergodic.
Problem: Estimate the dependence function

$$d_\alpha(\tau) := d_\alpha(X_\tau, X_0).$$

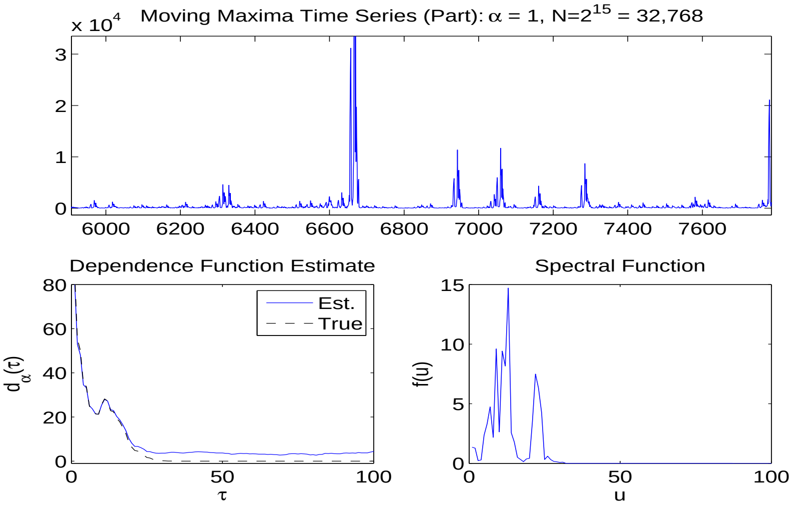
- A consistent estimator of the dependence function is:

$$\widehat{d}_{\alpha,p,n}(\tau) := c_{p,\alpha}\Big(\frac{2}{n}\sum_{k=1}^{n} X_k^p\Big)^{\alpha/p} - c_{p,\alpha}\Big(\frac{1}{n-\tau}\sum_{k=1}^{n-\tau}(X_{\tau+k}\vee X_k)^p\Big)^{\alpha/p},$$

where $0 < p < \alpha$ and $c_{p,\alpha} = \Gamma(1-p/\alpha)^{-\alpha/p}$.
- By ergodicity, for all $\gamma \in (0,1)$

$$\widehat{d}_{\alpha,p,n}(\tau) \stackrel{a.s.}{\to} d_\alpha(\tau) \quad \text{and} \quad \mathbb{E}|\widehat{d}_{\alpha,p,n}(\tau) - d_\alpha(\tau)|^\gamma \to 0, \quad \text{as } n\to\infty.$$

A numerical example



Concluding remarks

• Weintraub (1991) introduced 3 notions of "mixing" in terms of the de Haan's spectral representation.
○ No connections with the classical notions of ergodicity and mixing was established.
• We show that Weintraub's '0–mixing' is equivalent to mixing.
• Our work justifies and suggests a range of statistical methods for max–stable processes.
• Some old new tools on modeling and statistics for max–stable processes/fields?
• Further questions on:
○ estimation of the spectral function
○ representations of max–stable processes
○ random fields

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