On the structure of max–stable processes

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Preliminaries

Motivation: Consider i.i.d. processes $\{Y_t^{(i)}\}_{t\in T}$. If for some $a_n \sim n^{1/\alpha}\ell(n)$ $(\alpha > 0)$, we have

$$\Big\{\frac{1}{a_n}\bigvee_{1\le i\le n} Y_t^{(i)}\Big\}_{t\in T} \stackrel{f.d.d.}{\longrightarrow} \{X_t\}_{t\in T}, \quad (n\to\infty),$$

then $X = \{X_t\}_{t\in T}$ is max–stable ($\alpha-$Fréchet).

Def *The process $X = \{X_t\}_{t\in T}$ is max–stable ($\alpha-$Fréchet) if:*

$$\{X_t^{(1)} \vee \cdots \vee X_t^{(n)}\}_{t\in T} \stackrel{d}{=} n^{1/\alpha}\{X_t\}_{t\in T},$$

for all $n\in\mathbb{N}$, where $X^{(i)} = \{X_t^{(i)}\}_{t\in T}$ are independent copies of X and where $\alpha > 0$.

∘ The margins of X are then $\alpha-$Fréchet $(\alpha > 0)$, namely:

$$\mathbb{P}\{X_t \le x\} = \exp\{-\sigma_t^\alpha x^{-\alpha}\}, \quad x > 0,$$

with $\sigma_t^\alpha > 0$.

Fine print: For simplicity, we focus on max–stable processes with $\alpha-$Fréchet marginals. By transforming the margins, our theory applies to the most general definition of max–stable processes where the margins can be extreme value distributions of different types.

Spectral Representations

For an $\alpha-$Fréchet process $(\alpha > 0)$ $X = \{X_t\}_{t\in T}$, we have:

(a) de Haan's spectral representation:

$$X_t = \bigvee_{i=1}^{\infty} f_t(U_i)/\Gamma_i^{1/\alpha}, \quad (t \in T)$$

for a Poisson point process (Γ_i, U_i) on $(0,\infty) \times U$ with intensity $dx \times \mu(du)$.

(b) Extremal integrals:

$$X_t = {\int^{e}}_{U} f_t(u) M_\alpha(du), \quad (t \in T)$$

for an $\alpha-$Fréchet random *sup–measure* $M_\alpha(du)$ on (U, μ).

• The deterministic functions $f_t(u) \ge 0$ are called spectral functions of X and satisfy:

$$\int_U f_t(u)^\alpha \mu(du) < \infty, \quad (t \in T).$$

Fine print: The measure space (U, μ) can be chosen $([0,1], dx)$ if the process $\{X_t\}_{t\in T}$ is *separable in probability*, in particular, continuous in probability when T is a separable metric space.

Extremal integrals

Let M_α be a random $\alpha-$Fréchet sup–measure on (U,μ).

- For *simple functions* $f(u) = \sum_{i=1}^n a_i 1_{A_i}(u)$, $f(u) \geq 0$:

$$\int_U^e f(u) M_\alpha(du) = \bigvee_{1\le i\le n} a_i M_\alpha(A_i).$$

- The def of ${}^e\!\int_U f dM_\alpha$ extends to all $f \in L_+^\alpha(\mu)$ and

$$\mathbb{P}\{ \int_U^e f dM_\alpha \le x\} = \exp\{-\int_U f^\alpha d\mu x^{-\alpha}\}, \quad x > 0.$$

- For $f,\ g \in L_+^\alpha(\mu)$:

$$\int_U^e (af \vee bg) dM_\alpha = a \int_U^e f dM_\alpha \vee b \int_U^e g dM_\alpha \quad \text{(max–linearity)}$$

- ${}^e\!\int_U f dM_\alpha$ and ${}^e\!\int_U g dM_\alpha$ are *independent* if and only if $fg = 0$, (mod μ).

Benefits: For any $f_t \in L^\alpha_+(\mu)$, $t \in T$, we get a max–stable process:

$$X_t := {}^e\!\!\int_U f_t dM_\alpha$$

∘ For the finite–dimensional distributions, we have:

$$\begin{aligned}\mathbb{P}\{X_{t_i} \le x_i,\ 1 \le i \le d\} &= \mathbb{P}\{{}^e\!\!\int_U (\vee_{1\le i\le d} x_i^{-1} f_{t_i}) dM_\alpha \le 1\} \\ &= \exp\{-\int_U (\vee f_{t_i}^\alpha / x_i^\alpha) d\mu\}.\end{aligned}$$

Examples:

- (moving maxima) $X_t := {}^e\!\!\int_{\mathbb{R}} f(t-x) M_\alpha(dx)$, with $(U,\mu) \equiv (\mathbb{R}, dx)$ and $f \in L^\alpha_+(dx)$.
- (mixed moving maxima) With $(U,\mu) = (\mathbb{R}\times V, dx \times d\nu)$:

$$X_t := {}^e\!\!\int_{\mathbb{R}\times V} f(t-x, v) M_\alpha(dx, dv), \quad f(x,v) \in L^\alpha_+(dx, d\nu).$$

- (Brown–Resnick) With (U,μ) a probability space and $\{w_t(u)\}_{t\in\mathbb{R}}$ a standard Brownian motion on (U,μ):

$$X_t := {}^e\!\!\int_U e^{w_t(u) - |t|/2\alpha} M_\alpha(du), \quad t \in \mathbb{R}.$$

Max-linear isometries

Consider a max–stable process:

$$X_t = {}^{e}\!\!\int_U f_t dM_\alpha, \quad (t \in T).$$

Some Natural Questions:

- How does the structure of the $f_t(u)$'s determine the structure of $X = \{X_t\}_{t\in T}$ and vice versa?
- Given another representation $\{g_t\} \subset L_+^\alpha(V, \nu)$

$$\{X_t\}_{t\in T} \stackrel{d}{=} \Big\{ {}^{e}\!\!\int_V g_t d\widetilde{M}_\alpha \Big\},$$

what is the relationship between $\{f_t\}_{t\in T}$ and $\{g_t\}_{t\in T}$?

Answer: There exists a max–linear isometry

$$I : L_+^\alpha(U, \mu) \to L_+^\alpha(V, \nu),$$

such that

$$I(f_t) = g_t, \quad \text{for all } t \in T.$$

That is:

$$I(a_1 f_1 \vee a_2 f_2) = a_1 I(f_1) \vee a_2 I(f_2), \quad \text{for all } f_i \in L_+^\alpha(U,\mu),\ a_i \geq 0, \quad \text{(max–linear)}$$

and

$$\int_V I(f)^\alpha d\nu = \int_U f^\alpha d\mu \quad \text{(isometry)}.$$

◦ Conversely, any max–linear isometry $I : L_+^\alpha(U,\mu) \to L_+^\alpha(V,\nu)$ yields an equivalent spectral represenation $g_t := I(f_t)$ of the process X over (V,ν).

◦ Understanding the structure of the max–linear isometries is key!

• In Wang & S., 2009, we extend results of Hardin, 1981/82 on linear isometries to max–linear isometries.

• The role of these results in the theory of max–stable processes is best understood via the notion of a minimal representation.

Minimal spectral representations

Def *The spectral representation* $\{f_t\}_{t\in T} \subset L^\alpha_+(U,\mu)$ *of* X *is minimal if:*
(i) (full support) $\mathrm{supp}\{f_t,\ t\in T\} = U$ *(mod* μ*)*
(ii) (non–redundance) For any measurable $A \subset U$*, there exists*

$$B \in \rho\{f_t,\ t\in T\} \equiv \sigma\{f_t/f_s,\ t,s\in T\},$$

such that $\mu(A\Delta B) = 0$.

◦ This def is identical to the one of Rosiński (1995) in the sum–stable case and similar to Hardin (1982) and to the proper rep in de Haan and Pickands (1986).

- Why are minimal representations minimal?

◦ The full–support condition is natural.

◦ Consider the process

$$X_t := {\int^e}_{[0,1]} t^2 \sin^2(u) M_\alpha(du) = t^2 Z,$$

where $Z = {}^e\!\int_{[0,1]} \sin^2(u) M_\alpha(du)$.

- This representation is clearly redundant. Note that

$$\rho(f_t,\ t\in T) = \{\emptyset, [0,1]\} \not\approx \mathcal{B}_{[0,1]}.$$

◦ We have a natural, simpler representation:

$$X_t \stackrel{d}{=} {\int_U}^e f_t(u)\widetilde{M}_\alpha(du), \quad \text{with } f_t(u) = t^2,$$

and trivial $U = \{0\}$, and $\mu(du) = c\delta_0(du)$, $c := \int_{[0,1]} \sin^{2\alpha}(x)dx$.

• The ratio $\sigma-$algebra $\rho(f_t,\ t \in T)$ captures best the 'minimal information' needed to represent the process.

• What are the benefits of minimal representations?

Thm 1. (Wang & S.(2009)) *Let $\{f_t\}_{t\in T} \subset L^\alpha_+(\mu)$ be a minimal measurable rep of X. If $\{g_t\}_{t\in T} \subset L^\alpha_+(V,\nu)$ is another measurable rep of X, then:*

$$g_t(v) = h(v)f_t(\phi(v)), \quad \nu - a.e.$$

for some measurable $h \geq 0$ and $\phi : V \to U$. The map ϕ is unique (mod ν).
If $\{g_t\}$ is also minimal, then ϕ is bimeasurable, $\nu \sim \mu\circ\phi$ and

$$\frac{d\mu\circ\phi}{d\nu}(v) = h^\alpha(v) > 0.$$

Minimal representations with standardized support

Consider the sets $S_{I,N}$ where $I = 0,\ 1$ and $0 \le N \le \infty$:

- ◦ If $I = 1$, set $S_{1,N} = (0,1) \cup \{1, \cdots, N\}, \quad (0 \le N \le \infty)$
- ◦ If $I = 0$, set $S_{0,N} = \{1, \cdots, N\}, \quad (0 \le N \le \infty)$
- ◦ By convention: $S_{1,0} = (0,1),\ S_{0,\infty} = \mathbb{N}$ and $S_{0,0} = \emptyset$.
- Equip $S_{I,N}$ with the measure

$$\lambda_{I,N}(x) = dx + \sum_{i=1}^{N} \delta_{\{i\}}(dx).$$

Fine print: Every standard Lebesgue space is isomorphic to some $(S_{I,N}, \lambda_{I,N})$.

Def *A minimal representation $\{f_t\}_{t\in T} \subset L^{\alpha}_{+}(U,\mu)$ is said to have standardized support if, for some I, N: $(U,\mu) \equiv (S_{I,N}, \lambda_{I,N})$.*

Thm 2. (Wang & S., 2009) *Every separable in probability $\alpha-$Fréchet process X has a minimal representation with standardized support:*

$$\{X_t\}_{t\in T} \stackrel{d}{=} \Big\{ {}^{e}\!\!\int_{S_{I,N}} f_t dM_\alpha \Big\}_{t\in T}.$$

Continuous–Discrete Decomposition

Consider an $\alpha-$Fréchet process X with the minimal rep of standardized support:

$$\{X_t\}_{t\in T} \stackrel{d}{=} \left\{ \int^{e}_{S_{I,N}} f_t dM_\alpha \right\}_{t\in T}.$$

By setting

$$X_t^I := \int^{e}_{S_{I,N}\cap(0,1)} f_t dM_\alpha \quad \text{and} \quad X_t^N := \int^{e}_{S_{I,N}\cap\mathbb{N}} f_t dM_\alpha,$$

we obtain the continuous–discrete decomposition:

$$\{X_t\}_{t\in T} \stackrel{d}{=} \{X_t^I \vee X_t^N\}_{t\in T}.$$

The components X_t^I and X_t^N are independent.

Intuition: Suppose $I = 1$ and $N > 0$. Then,

$$X_t^I = \int^{e}_{(0,1)} f_t dM_\alpha \quad \text{and} \quad X_t^N = \bigvee_{i=1}^{N} f_t(i) Z_i,$$

where $Z_i = M_\alpha\{i\}$ are i.i.d. standard $\alpha-$Fréchet, independent of X_t^I.

∘ $\{X_t^I\}$ is the continuous and $\{X_t^N\}_{t\in T}$ the discrete spectral components of X.

Fine print: One of the compponents vanishes if $N = 0$ or $I = 0$.

• The continous–discerete decomposition does not depend on the choice of the representation.

Thm (Wang & S., 2009) *Let* $\{g_t\}_{t\in T} \subset L_+^\alpha(S_{I',N'}, \lambda_{I',N'})$ *be another minimal rep of* X *with standardized support, then* $(I, N) \equiv (I', N')$,

$$\{X_t^I\} \stackrel{d}{=} \{X_t^{I'}\}_{t\in T} \quad \textit{and} \quad \{X_t^N\} \stackrel{d}{=} \{X_t^{N'}\}_{t\in T},$$

where $X_t^{I'} := {}^e\!\!\int_{S_{I,N}\cap(0,1)} g_t dM_\alpha$ *and* $X_t^{N'} = {}^e\!\!\int_{S_{I,N}\cap\mathbb{N}} d_t dM_\alpha$.

• Moreover, for the discrete component, we have that:

$$\{X_t^N\}_{t\in T} \stackrel{d}{=} \Big\{ \bigvee_{i=1}^N \phi_t(i) Z_i \Big\}_{t\in T},$$

for some unique set of functions $\{\phi_t(i),\ 1 \le i \le N\}$.

Discrete Principal Components

Consider the spectrally discrete component of the process $\{X_t\}_{t\in T}$:

$$X_t^N = \bigvee_{i=1}^{N} \phi_t(i) Z_i, \quad (t \in T),$$

for i.i.d. standard $\alpha-$Fréchet Z_i's.

• The functions $t \mapsto \phi_t(i),\ 1 \le i \le N$ are unique up to permutation of the indices.

◦ The $\phi_t(i)$'s are the discrete principal components of X.

• Not all sequences of non–negative of functions can be discrete principal components.

Fine print: Prop: (Wang & S., 2009) *A countable set of functions* $\phi := \{\phi_t(i) \ge 0,\ 1 \le i \le N\}$ *can be discrete principal components of an* $\alpha-$*Fréchet process if and only if,* ϕ *is a minimal representation. Namely, if (i)* $\sum_{i=1}^N \phi_t^\alpha(i) < \infty$ *and (ii)* $\rho(\phi_t(\cdot),\ t \in T) = 2^{\{1,\cdots,N\}}$.

Discrete Principal Components: Applications and Implications

Applications: Given a spectrally discrete statistical model, estimate:

- ◦ The order N, if finite.
- ◦ The (unique) principal component functions $t \mapsto \phi_t(i)$, for $1 \le i \le N$.

Thm (Wang & S., 2009) *Let $\{X_t\}_{t\in\mathbb{R}}$ be a measurable and stationary $\alpha-$Fréchet process. Then, the spectrally discrete component of X is either zero or trivial, i.e.*

$$\{X_t^N\}_{t\in\mathbb{R}} \stackrel{d}{=} \{Z\}_{t\in\mathbb{R}},$$

for some random variable Z.

- Certainly, with i.i.d. $\alpha-$Fréchet Z_i's:

$$X_t := \bigvee_{i\in\mathbb{Z}} f(t-i)Z_i, \quad (t \in \mathbb{Z}),$$

is a non–trivial stationary and spectrally discrete processes.

Co–spectral functions

Let now T be a separable metric space, equipped with a Borel measure λ. Consider the $\alpha-$Fréchet process $X = \{X_t\}_{t\in T}$:

$$X_t := \int^e_U f(t,u) M_\alpha(du), \quad (t \in T),$$

where $(t,u) \mapsto f(t,u)$ is measurable.

• Focus on the co–spectral functions:

$$t \mapsto f(t,u) \in L^0_+(T,\lambda), \qquad \text{for fixed } u \in U.$$

• Can show that the co–spectral functions of X do not depend on the representation (up to rescaling)!

Fine print: If (U,μ) is standard Lebesgue, then X is measurable, if and only if, $(t,u) \mapsto f(t,u)$ has a measurable modification.

Co–spectral functions: Classification

Let $\mathcal{P}$ be a positive (measurable) cone in $L^0_+(T,\lambda)$, i.e. $c\mathcal{P} \subset \mathcal{P}$. Consider the partition of $U = A \cup B$ into disjoint components:

$$A := \{u \in U \,:\, f(\cdot,u) \in \mathcal{P}\} \quad \text{and} \quad B := U \setminus A = \{u \in U \,:\, f(\cdot,u) \notin \mathcal{P}\}.$$

This yields the decomposition:

$$\{X_t\}_{t\in T} \stackrel{d}{=} \{X_t^A \vee X_t^B\}_{t\in T}, \tag{1}$$

where

$$X_t^A := \int^e_A f(t,u) M_\alpha(du) \quad \text{and} \quad X_t^B := \int^e_B f(t,u) M_\alpha(du)$$

are two independent processes.

• The decomposition (1) does not depend on the choice of the measurable rep $\{f(t,u)\}_{(t,u)\in T\times U}$.

Idea of proof: WLOG suppose that $\{f(t,u)\}_{t\in T}$ is minimal and let $\{g(t,v)\}_{t\in T} \subset L^\alpha_+(V,\nu)$ is another measurable rep of $X = \{X_t\}_{t\in T}$.

Then, by Thm 1:

$$g(t,v) = h(v) f(t, \phi(v)), \quad \text{where } h(v) \geq 0.$$

Since $\mathcal{P}$ is a cone,

$$g(\cdot, v) \in \mathcal{P} \quad \Leftrightarrow \quad f(\cdot, \phi(v)) \in \mathcal{P},$$

which shows that the corresponding partition of V is:

$$V = \widetilde{A} \cup \widetilde{B} := \phi^{-1}(A) \cup \phi^{-1}(B)$$

A change of variables, yields:

$$\Big\{ {\int_A}^{e} f_t dM_\alpha \Big\}_{t \in T} \stackrel{d}{=} \Big\{ {\int_{\widetilde{A}}}^{e} g_t d\widetilde{M}_\alpha \Big\}_{t \in T},$$

completing the proof.

Applications

Corollary: *Let*

$$\Big\{ {}^{e}\!\!\int_U f(t,u) M_\alpha(du) \Big\} \stackrel{d}{=} \Big\{ {}^{e}\!\!\int_V g(t,v) \widetilde{M}_\alpha(dv) \Big\}.$$

Then, given a cone $\mathcal{P} \subset L^0_+(T)$,

$$f(\cdot, u) \in \mathcal{P}, \text{a.e.} \quad \text{if and only if} \quad g(\cdot, v) \in \mathcal{P}, \text{a.e.}$$

∘ Let $(U, \mu) \equiv (\mathbb{R}^d, dx)$ and $f,\ g \in L^\alpha_+(\mathbb{R}^d)$. Consider the moving maxima random fields

$$X_t := {}^{e}\!\!\int_{\mathbb{R}^d} f(t-u) M_\alpha(du) \quad \text{and} \quad Y_t := {}^{e}\!\!\int_{\mathbb{R}^d} g(t-u) M_\alpha(du).$$

Then,

$$\{X_t\}_{t\in\mathbb{R}^d} \stackrel{d}{=} \{Y_t\}_{t\in\mathbb{R}^d},$$

if and only if, for some $\tau \in \mathbb{R}^d$

$$g(\cdot) = f(\cdot + \tau).$$

Key References

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