Supplementary Material for Bayesian Nonparametric Multilevel Clustering with Contexts

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This note provides supplementary information for the main paper. It has three parts: a) the proof for the marginalization property of our proposed model, b) detailed derivations for our inference, and c) equations to show how the perplexity in the experiment was computed.

1 Proof for Marginalization Property (Theorem 4)

We start with a proposition on the marginalization result for DPM with the product measure then move on the final proof for our proposed model.

1.1 Marginalization of DPM with product measure

Let $H$ be a measure over some measurable space $(\Theta, \Sigma)$. Let $P$ be the set of all measures over $(\Theta, \Sigma)$, suitably endowed with some $\sigma$-algebra. Let $G \sim DP(\alpha H)$ be a draw from a Dirichlet process.

Lemma 1. Let $S_1\ldots S_n$ be $n$ measurable sets in $\Sigma$. We form a measurable partition of $\Theta$, a collection of disjoint measurable sets, that generate $S_1,\ldots,S_n$ as follows. If $S$ is a set, let $S_1=S\setminus S$. Then $S^* = \{\bigcap_{i=1}^n S_i^c | c_i \in \{1,-1\}\}$ is a partition of $\Theta$ into a finite collection of disjoint measurable sets with the property that any $S_i$ can be written as a union of some sets in $S^*$. Let the element of $S^*$ be $A_1\ldots A_{n^*}$ (note $n^* \leq 2^n$). Then the expectation

$$E_G[G(S_1),\ldots,G(S_n)] = \int \prod_{i=1}^n G(S_i) \, DP(dG | \alpha H)$$

depends only on $\alpha$ and $H(A_i)$. In other words, the above expectation can be written as a function $E_n(\alpha, H(A_1),\ldots,H(A_{n^*}))$.

It is easy to see that since $S_i$ can always be expressed as the sum of some disjoint sets $A_i$, $G(S_i)$ can respectively be written as the sum of some $G(A_i)$. Furthermore, by definition of a Dirichlet process, the vector $(G(A_1),\ldots,G(A_{n^*}))$ distributed according to a finite Dirichlet distribution $(\alpha H(A_1),\ldots,\alpha H(A_{n^*}))$, therefore the expectation $E_G[G(S_i)]$ depends only on $\alpha$ and $H(A_i)$ (s).

Definition 2. (DPM) A DPM is a probability measure over $\Theta^n \ni (\theta_1,\ldots,\theta_n)$ with the usual product sigma algebra $\Sigma^n$ such that for every collection of measurable sets $\{(S_1,\ldots,S_n) : S_i \in \Sigma, i = 1,\ldots,n\}$:

$$DPM(\theta_1 \in S_1,\ldots,\theta_n \in S_n | \alpha, H) = \int \prod_{i=1}^n G(S_i) \, DP(dG | \alpha H)$$
Consider two measurable spaces $(\Theta_1, \Sigma_1)$ and $(\Theta_2, \Sigma_2)$ and let $(\Theta, \Sigma)$ be their product space where $\Theta = \Theta_1 \times \Theta_2$ and $\Sigma = \Sigma_1 \times \Sigma_2$. We present the general theorem that states the marginal result from a product base measure.

**Proposition 3.** Let $H^*$ be a measure over the product space $\Theta = \Theta_1 \times \Theta_2$. Let $H_1$ be the marginal of $H^*$ over $\Theta_1$ in the sense that for any measurable set $A \in \Sigma_1$, $H_1(A) = H^*(A \times \Theta_2)$. Then:

\[DPM\left(\theta_1^{(1)} \in S_1, \ldots, \theta_n^{(1)} \in S_n \mid \alpha, H_1\right) = DPM\left(\left(\theta_1^{(1)}, \theta_1^{(2)}\right) \in S_1 \times \Theta_2, \ldots, \left(\theta_n^{(1)}, \theta_n^{(2)}\right) \in S_n \times \Theta_2 \mid \alpha, H^*\right)\]

for every collection of measurable sets \(\{(S_1, \ldots, S_n) : S_i \in \Sigma_1, i = 1, \ldots, n\}\).

**Proof.** Since \(\{(S_1, \ldots, S_n) : S_i \in \Sigma_1, i = 1, \ldots, n\}\) are rectangles, expanding the RHS using Definition 2 gives:

\[\text{RHS} = \int G(S_1 \times \Theta_2) \ldots G(S_n \times \Theta_2) dDP(dG|\alpha, H^*)\]

Let $T_i = S_i \times \Theta_2$, the above expression is the expectation of $\prod_i G(T_i)$ when $G \sim DP(\alpha H^*)$. Forming collection of the disjoint measurable sets $T^* = (B_1 \ldots B_n)$ that generates $T_i$, then note that $B_i = A_i \times \Theta_2$, and $S^* = (A_1 \ldots A_n)$ generates $S_i$. By definition of $H_1$, $H_1(A_i) = H^*(A_i \times \Theta_2) = H^*(B_i)$. Using the Lemma 1 above, \(\text{RHS} = E_n(\alpha, H^*(B_1) \ldots H^*(B_n))\), while \(\text{LHS} = E_n(\alpha, H_1(A_1) \ldots H_1(A_n))\) and they are indeed the same. \(\square\)

We note that $H^*$ can be any arbitrary measure on $\Theta$ and, in general, we do not require $H^*$ to factorize as product measure.

### 1.2 Marginalization result for our proposed model

Recall that we are considering a product base-measure of the form $H^* = H \times DP(vQ_0)$ for the group-level DP mixture. Drawing from a DP mixture with this base measure, each realization is a pair $(\theta_j, Q_j)$: $\theta_j$ is then used to generate the context $x_j$ and $Q_j$ is used to repeatedly generate the set of content observations $w_{ji}$ within the group $j$. Specifically,

\[U \sim DP(\alpha(H \times DP(vQ_0))) \quad \text{where} \quad Q_0 \sim DP(\eta S)\]

\[(\theta_j, Q_j) \overset{iid}{\sim} U \quad \text{for} \quad j = 1, \ldots, J \tag{3}\]

\[\varphi_{ji} \overset{iid}{\sim} Q_j, \quad \text{for each } j \text{ and } i = 1, \ldots, N_j\]

In the above, $H$ and $S$ are respectively base measures for context and content parameters $\theta_j$ and $\varphi_{ji}$. We start with a definition for nested Dirichlet Process Mixture (nDPM) to proceed further.

**Definition 4.** (nested DP mixture) A nDPM is a probability measure over $\Theta^J \times \prod_{j=1}^{J} N_j$ equipped with the usual product sigma algebra $\Sigma^{N_1} \times \cdots \times \Sigma^{N_J}$ such that for any collection of measurable sets \(\{(S_{ji}) : S_{ji} \in \Sigma, j = 1, \ldots, J, i = 1, \ldots, N_j\}\):

\[\text{nDPM}(\varphi_{11} \in S_{1}^{(1)}, \ldots, \varphi_{1N_1} \in S_{n_1}^{(1)}, \ldots, \varphi_{J1} \in S_{1}^{(j)}, \ldots, \varphi_{JN_J} \in S_{N_J}^{(j)} | \alpha, v, \eta, S) = \int \left\{ \prod_{j=1}^{J} \prod_{i=1}^{N_j} Q_j \left( S_{ij}^{(j)} \right) U(dQ_j) \right\} \text{DP}(dU | \alpha \text{DP}(vQ_0)) \text{DP}(dQ_0 | \eta, S)\]

We now state the main marginalization result for our proposed model.

**Theorem 5.** Given $\alpha, H$ and $\alpha, v, \eta, S$, let $\theta = (\theta_j : \forall j)$ and $\varphi = (\varphi_{ji} : \forall j, i)$ be generated as in Eq (3). Then, marginalizing out $\varphi$ results in $DPM(\theta | \alpha, H)$, whereas marginalizing out $\theta$ results in $\text{nDPM}(\varphi | \alpha, v, \eta, S)$.

**Proof.** First we make observation that if we can show Proposition 3 still holds when $H_1$ is random with $H_2$ is fixed and vice versa, then the proof required is an immediate corollary of Proposition 3 by letting $H^* = H_1 \times H_2$ where we first let $H_1 = H, H_2 = DP(vQ_0)$ to obtain the proof for the first result, and then swap the order $H_1 = DP(vQ_0), H_2 = H$ to get the second result.
To see that Proposition 3 still holds when $H_2$ is a random measure and $H_1$ is fixed, we let the product base measure $H^* = H_1 \times H_2$ and further let $\mu$ be a prior probability measure for $H_2$, i.e., $H_2 \sim \mu(\cdot)$. Consider the marginalization over $H_2$:

$$\int_{H_2} \text{DPM} \left( \left( \theta_{1}^{(1)}, \theta_{1}^{(2)} \right) \in S_{1} \times \Theta_{2}, \ldots, \left( \theta_{n}^{(1)}, \theta_{n}^{(2)} \right) \in S_{n} \times \Theta_{2} \mid \alpha, H^* \right) \mu \left( H_2 \right)$$

$$= \int_{\Sigma_{2}} \text{DPM} \left( \theta_{1}^{(1)} \in S_{1}, \ldots, \theta_{n}^{(1)} \in S_{n} \mid \alpha, H_{1} \right) \mu \left( H_2 \right)$$

$$= \text{DPM} \left( \theta_{1}^{(1)} \in S_{1}, \ldots, \theta_{n}^{(1)} \in S_{n} \mid \alpha, H_{1} \right) \int_{\Sigma_{2}} \mu \left( H_2 \right)$$

$$= \text{DPM} \left( \theta_{1}^{(1)} \in S_{1}, \ldots, \theta_{n}^{(1)} \in S_{n} \mid \alpha, H_{1} \right)$$

When $H_1$ is random and $H_2$ is fixed. Let $\lambda(\cdot)$ be a prior probability measure for $H_1$, i.e., $H_1 \sim \lambda(\cdot)$. It is clear that Proposition 3 holds for each draw $H_1$ from $\lambda(\cdot)$. This complete our proof.

1.3 Additional result for correlation analysis in nDPM

We now consider the correlation between $\varphi_{ik}$ and $\varphi_{jk'}$ for arbitrary $i, j, k$ and $k'$, i.e., we need to evaluate:

$$P \left( \varphi_{ik} \in A_1, \varphi_{jk'} \in A_2 \mid \alpha, \eta, v, S \right)$$

for two measurable sets $A_{1}, A_{2} \in \Sigma$ by integrating out over all immediate random measures. We use an explicit stick-breaking representation for $U$ where $U \sim \text{DP} \left( \alpha \text{DP} \left( vQ_{0} \right) \right)$ as follows

$$U = \sum_{k=1}^{\infty} \pi_{k} \delta_{Q_{k}^{*}}$$

(4)

where $\pi \sim \text{GEM} (\alpha)$ and $Q_{k}^{*} \overset{\text{iid}}{\sim} \text{DP} \left( vQ_{0} \right)$. We use the notation $\delta_{Q_{k}^{*}}$ to denote the atomic measure on measure, placing its mass at measure $Q_{k}^{*}$.

For $i = j$, we have:

$$P \left( \varphi_{ik} \in A_1, \varphi_{jk'} \in A_2 \mid Q_{1}, \ldots, Q_{j} \right) = Q_{i} \left( A_{1} \right) Q_{i} \left( A_{2} \right)$$

Sequentially take expectation over $Q_i$ and $U$:

$$\int_{Q_{i}} Q_{i} \left( A_{1} \right) Q_{i} \left( A_{2} \right) dU \left( Q_{i} \right) = \int_{Q_{i}} Q_{i} \left( A_{1} \right) Q_{i} \left( A_{2} \right) d \left( \sum_{k=1}^{\infty} \pi_{k} \delta_{Q_{k}^{*}} \right)$$

$$= \sum_{k} \pi_{k} \left[ Q_{k}^{*} \left( A_{1} \right) Q_{k}^{*} \left( A_{2} \right) \right]$$

$$\int_{U} \sum_{k=1}^{\infty} \pi_{k} \left[ Q_{k}^{*} \left( A_{1} \right) Q_{k}^{*} \left( A_{2} \right) \right] d\text{DP} \left( U \mid \alpha \text{DP} \left( vQ_{0} \right) \right) = \mathbb{E} \left\{ \sum_{k} \pi_{k} \left[ Q_{k}^{*} \left( A_{1} \right) Q_{k}^{*} \left( A_{2} \right) \right] \right\}$$

$$= \sum_{k} \mathbb{E} \left[ \pi_{k} \right] \mathbb{E} \left[ Q_{k}^{*} \left( A_{1} \right) Q_{k}^{*} \left( A_{2} \right) \right]$$

$$= \frac{Q_{0} \left( A_{1} \cap A_{2} \right) + Q_{0} \left( A_{1} \right) Q_{0} \left( A_{2} \right)}{v \left( v + 1 \right)} \left( \sum_{k} \mathbb{E} \left[ \pi_{k} \right] \right)$$

$$= \frac{Q_{0} \left( A_{1} \cap A_{2} \right) + Q_{0} \left( A_{1} \right) Q_{0} \left( A_{2} \right)}{v \left( v + 1 \right)}$$

Integrating out $Q_{0} \sim \text{DP} \left( vS \right)$ we get:
\[ P (\varphi_{ik} \in A_1, \varphi_{jk'} \in A_2 \mid \alpha, v, \eta, S) = E_{Q_0, \eta, S} \left[ \frac{Q_0 (A_1 \cap A_2) + Q_0 (A_1) Q_0 (A_2)}{v(v+1)} \right] \]
\[ = \frac{1}{v(v+1)} \left\{ S (A_1 \cap A_2) + \frac{S (A_1 \cap A_2) + S (A_1) S (A_2)}{\eta(\eta+1)} \right\} \]
\[ = \frac{S (A_1 \cap A_2)}{v(v+1)} + \frac{S (A_1 \cap A_2) + S (A_1) S (A_2)}{v(v+1)\eta(\eta+1)} \]

For \( i \neq j \), since \( Q_i \) and \( Q_j \) are conditionally independent given \( U \), we get:
\[ P (\varphi_{ik} \in A_1, \varphi_{jk'} \in A_2 \mid Q_1, \ldots, Q_j) = Q_i (A_1) Q_j (A_2) \]

Let \( a_k = Q_k^* (A_1) \), \( b_k = Q_k^* (A_2) \) and using Definition (4), integrating out \( U \) conditional on \( Q_0 \) with the stick-breaking representation in Eq (4):
\[ P (\varphi_{ik} \in A_1, \varphi_{jk'} \in A_2 \mid vQ_0) = \left( \int_U Q_i (A_1) dU \right) \left( \int_U Q_j (A_2) dU \right) \]
\[ = E \left[ \sum_k \pi_k Q_k^* (A_1) \right] \left[ \sum_{k'} \pi_{k'} Q_{k'}^* (A_2) \right] \]
\[ = E (\pi_1 a_1 + \pi_2 a_2 + \ldots) (\pi_1 b_1 + \pi_2 b_2 + \ldots) \]
\[ = E \left( \sum_k \pi_k^2 a_k b_k \right) + E \left( \sum_{k' \neq k} \pi_k \pi_{k'} a_k b_{k'} \right) \]
\[ = A E \left( \sum_k \pi_k^2 \right) + B E \left( \sum_{k' \neq k} \pi_k \pi_{k'} \right) \]
\[ = A \sum_k E [\pi_k^2] + B \left( 1 - \sum_k E [\pi_k^2] \right) \]

where
\[ A = E [a_k b_k] = E [Q_k^* (A_1) Q_k^* (A_2)] = \frac{Q_0 (A_1 \cap A_2) + Q_0 (A_1) Q_0 (A_2)}{v(v+1)} \]

and since \( Q_k^* (s) \) are iid draw from \( DP (vQ_0) \) we have:
\[ B = E [a_k b_{k'}] = E [Q_k^* (A_1) Q_{k'}^* (A_2)] = E [Q_k^* (A_1)] E [Q_{k'}^* (A_2)] \]
\[ = Q_0 (A_1) Q_0 (A_2) \]

Lastly, since \( (\pi_1, \pi_2, \ldots) \sim GEM (\alpha) \), using the property of its stick-breaking representation \( \sum_k E [\pi_k^2] = \frac{1}{1+\alpha} \). Put things together we obtain the expression for the correlation of \( \varphi_{ik} \) and \( \varphi_{jk'} \) for \( i \neq j \) conditional on \( Q_0 \) as:
\[ P (\varphi_{ik} \in A_1, \varphi_{jk'} \in A_2 \mid vQ_0) = \frac{Q_0 (A_1 \cap A_2) + Q_0 (A_1) Q_0 (A_2)}{(1+\alpha) v(v+1)} + \frac{\alpha}{1+\alpha} Q_0 (A_1) Q_0 (A_2) \]
\[ = \frac{Q_0 (A_1 \cap A_2)}{(1+\alpha) v(v+1)} + \frac{\alpha v(v+1)+1}{(1+\alpha) v(v+1)} Q_0 (A_1) Q_0 (A_2) \]

Next, integrating out \( Q_0 \sim DP (vS) \) we get:
\[ P (\varphi_{ik} \in A_1, \varphi_{jk'} \in A_2 \mid \alpha, v, \eta, S) = \frac{\alpha v(v+1)+1}{(1+\alpha) v(v+1)} E [Q_0 (A_1) Q_0 (A_2)] + \frac{E [Q_0 (A_1 \cap A_2)]}{(1+\alpha) v(v+1)} \]
\[ = \frac{\alpha v(v+1)+1}{(1+\alpha) v(v+1)} S (A_1 \cap A_2) + \frac{S (A_1 \cap A_2)}{\eta(\eta+1)} \]
2 Model Inference Derivations

We provide detailed derivations for model inference with the graphical model displayed in Fig 1. The variables $\phi_k$, $\psi_m$, $\pi$, $\tau_k$ are integrated out due to conjugacy property. We need to sample these latent variables $z, l, \epsilon$ and hyper parameters $\alpha, v, \eta$. For convenience of notation, we denote $z_{-j}$ is a set of latent context variable $z$ in all documents excluding document $j$, $l_{ji}$ is all of hidden variables $l_{ji}$ in document $j$, and $l_{-j*}$ is all of $l$ in other documents rather than document $j$-th.

Sampling $z$

Sampling context index $z_j$ needs to take into account the influence of the corresponding context topics:

$$p(z_j = k \mid z_{-j}, l, x, \alpha, H) \propto \underbrace{p(z_j = k \mid z_{-j}, \alpha)}_{\text{CRP for context topic}} \cdot \underbrace{p(x_j \mid z_j = k, z_{-j}, x_{-j}, H)}_{\text{context predictive likelihood}} \times \underbrace{p(l_{j*} \mid z_j = k, l_{-j*}, z_{-j}, \epsilon, v)}_{\text{content latent marginal likelihood}}$$

The first term can easily be recognized as a form of Chinese Restaurant Process (CRP):

$$p(z_j = k \mid z_{-j}, \alpha) = \begin{cases} \frac{n^k_{-j}}{n^*_{-j} + \alpha} & \text{if } k \text{ is previously used} \\ \frac{\alpha}{n^*_{-j} + \alpha} & \text{if } k \text{ is new} \end{cases}$$

where $n^k_{-j}$ is the number of data $z_j = k$ excluding $z_j$, and $n^*_{-j}$ is the count of all $z$, except $z_j$.

The second expression is the predictive likelihood from the context observations under the context component $\phi_k$. Specifically, let $f(\cdot \mid \phi)$ and $h(\cdot)$ be respectively the density function for $F(\phi)$ and $H$, the conjugacy between $F$ and $H$ allows us to integrate out the mixture component parameter $\phi_k$, leaving us the conditional density of $x_j$ under the mixture component $k$ given all the context data items exclude $x_j$:

$$p(x_j \mid z_j = k, z_{-j}, x_{-j}, H) = \frac{\int_{\phi_k} f(x_j \mid \phi_k) \prod_{j' \neq j, z_{j'} = k} f(x_{j'} \mid \phi_k) h(\phi_k) d\phi_k}{\int_{\phi_k} \prod_{j' \neq j, z_{j'} = k} f(x_{j'} \mid \phi_k) h(\phi_k) d\phi_k} = f^{-x_j}_{k} (x_j)$$
Finally, the last term is the contribution from the multiple latent variables of corresponding topics to that context. Since $l_{ji} \mid z_j = k$ iid $\text{Mult}(\tau_k)$ where $\tau_k \sim \text{Dir}(\epsilon_1, \ldots, \epsilon_M, \epsilon_{\text{new}})$, we shall attempt to integrate out $\tau_k$. Using the Multinomial-Dirichlet conjugacy property we proceed to compute the last term in Eq. (5) as following:

$$p(l_{js} \mid z_j = k, z_{-j}, l_{-js}, \epsilon, v) = \int_{\tau_k} p(l_{js} \mid \tau_k) \times p(\tau_k \mid \{l_{js}' \mid z_{j'} = k, j' \neq j\}, \epsilon, v) \, d\tau_k \tag{6}$$

Recognizing the term $p(\tau_k \mid \{l_{js}' \mid z_{j'} = k, j' \neq j\}, \epsilon, v)$ is a posterior density, it is Dirichlet-distributed with the updated parameters

$$p(\tau_k \mid \{l_{js}' \mid z_{j'} = k, j' \neq j\}) = \text{Dir}(\epsilon_1 + c_{k,1}^{-j}, \ldots, \epsilon_M + c_{k,M}^{-j}, \epsilon_{\text{new}}) \tag{7}$$

where $c_{k,m}^{-j} = \sum_{j' \neq j}^{N_j} (l_{j'i} = m, z_{j'} = k)$ is the count of topic $m$ being assigned to context $k$ excluding document $j$. Using this result, $p(l_{js} \mid \tau_k)$ is a predictive likelihood for $l_{js}$ under the posterior Dirichlet parameters $\tau_k$ in Eq 7 and therefore can be evaluated to be:

$$p(l_{js} \mid z_j = k, z_{-j}, l_{-js}, \epsilon, v) = \int_{\tau_k} p(l_{js} \mid \tau_k) \times \text{Dir}(\epsilon_1 + c_{k,1}^{-j}, \ldots, \epsilon_M + c_{k,M}^{-j}, \epsilon_{\text{new}}) \, d\tau_k$$

$$= \int_{\tau_k} \prod_{m=1}^{M} \tau_{k,m}^{c_{k,m}^-} \frac{\prod_{m=1}^{M} (\sum_{m=1}^{M} (\epsilon_m + c_{k,m}^-))}{\prod_{m=1}^{M} \Gamma(\epsilon_m + c_{k,m}^-)} \times \prod_{m=1}^{M} \frac{\epsilon_m + c_{k,m}^- + c_{k,m}^+}{\tau_{k,m}^{c_{k,m}^- + c_{k,m}^+}} \, d\tau_k$$

$$= \left\{ \begin{array}{ll}
A = \frac{\prod_{m=1}^{M} (\sum_{m=1}^{M} \epsilon_m + c_{k,m}^-)}{\prod_{m=1}^{M} \Gamma(\sum_{m=1}^{M} \epsilon_m + c_{k,m}^-)} \times \prod_{m=1}^{M} \frac{\epsilon_m + c_{k,m}^- + c_{k,m}^+}{\Gamma(\epsilon_m + c_{k,m}^-)} & \text{if } k \text{ previously used} \\
B = \frac{\prod_{m=1}^{M} \epsilon_m}{\prod_{m=1}^{M} \Gamma(\sum_{m=1}^{M} \epsilon_m + c_{k,m}^-)} & \text{if } k = \epsilon_{\text{new}}
\end{array} \right.$$ 

note that $\epsilon = (\epsilon_1, \epsilon_2, \ldots, \epsilon_M, \epsilon_{\text{new}})$, here $\epsilon_{1:M} = (\epsilon_1, \epsilon_2, \ldots, \epsilon_M)$, when sampling $z_j$ we only use $M$ active components from the previous iteration. In summary, the conditional distribution to sample $z_j$ is given as:

$$p(z_j = k \mid z_{-j}, l, x, \alpha, H) \propto \left\{ \begin{array}{ll}
\nu_{k,j}^{z_j} \times \int_k^{\epsilon_{\text{new}}} (x_j) \times A & \text{if } k \text{ previously used} \\
\alpha \times \int_{\epsilon_{\text{new}}}^{x_{ji}} (x_{ji}) \times B & \text{if } k = \epsilon_{\text{new}}
\end{array} \right.$$ 

Implementation note: to evaluate $A$ and $B$, we make use of the marginal likelihood resulted from a Multinomial-Dirichlet conjugacy.

**Sampling $l$**

Let $w_{-ji}$ be the same set as $w$ excluding $w_{ji}$, i.e $w_{-ji} = \{w_{uv} : u \neq j \cap v \neq i\}$, then we can write

$$p(l_{ji} = m \mid l_{-ji}, z_j = k, v, w, S) \propto p(w_{ji} \mid w_{-ji}, l_{ji} = m, \rho) \times p(l_{ji} = m \mid l_{-ji}, z_j = k, \epsilon_m, v) \tag{8}$$

*content predictive likelihood* *CRF for content topic*
The first argument is computed as log likelihood predictive of the content with the component $\psi_m$

$$p(w_{ji} \mid w_{-ji}, l_{ji} = m, \rho) = \frac{\int_{\lambda_m} s(w_{ji} \mid \lambda_m) \left[ \prod_{u \notin w_{-ji}(m)} y(u \mid \lambda_m) \right] s(\lambda_m) d\lambda_m}{\int_{\lambda_m} \left[ \prod_{u \notin w_{-ji}(m)} y(u \mid \lambda_m) \right] s(\lambda_m) d\lambda_m}$$ (9)

And the second term is inspired by Chinese Restaurant Franchise (CRF) as:

$$p(l_{ji} = m \mid l_{-ji}, \epsilon_m, v) = \begin{cases} c_{k,m} + v\epsilon_m & \text{if mis used previously} \\ v\epsilon_{new} & \text{if } m = m_{new} \end{cases}$$ (10)

where $c_{k,m}$ is the number of data point $\{|l_{ji}| l_{ji} = m, z_j = k, 1 \leq j \leq J, 1 \leq i \leq N_j \}$. The final form to sample $l_{ji}$ is given as:

$$p(l_{ji} = m \mid l_{-ji}, z_j = k, w, v, \epsilon) \propto \begin{cases} (c_{k,m} + v\epsilon_m) \times y_m^{-w_{ji}}(w_{ji}) & \text{if mis used previously} \\ v\epsilon_{new} \times y_m^{-w_{ji}}(w_{ji}) & \text{if } m = m_{new} \end{cases}$$

### Sampling $\epsilon$

Note that sampling $\epsilon$ require both $z$ and $l$.

$$p(\epsilon \mid l, z, v, \eta) \propto p(l \mid \epsilon, v, z, \eta) \times p(\epsilon \mid \eta)$$ (11)

Isolating the content variables $l_{ji}^k$ generated by the same context $z_j = k$ into one group $l_j^k = \{l_{ji} : 1 \leq i \leq N_j, z_j = k\}$ the first term of 11 can be expressed following:

$$p(l \mid \epsilon, v, z, \eta) = \prod_{k=1}^{K} \int p\left(l_{ji}^k \mid \tau_k\right) p\left(\tau_k \mid \epsilon\right) d\tau_k = \prod_{k=1}^{K} \frac{\Gamma(v)}{\Gamma(v + n_{ks})} \prod_{m=1}^{M} \frac{\Gamma(v\epsilon_m + n_{km})}{\Gamma(v\epsilon_m)}$$

where $n_{ks} = |\{w_{ji} \mid z_j = k, i = 1, ... N_j\}|$ and $n_{km} = |\{w_{ji} \mid z_j = k, l_{ji} = m, 1 \leq j \leq J, 1 \leq i \leq N_j, \}|$.

Let $\eta_r = \frac{\eta}{M}$, $\eta_{new} = \frac{R - M}{R} \eta$ and recall that $\epsilon \sim \text{Dir}(\eta_r, ..., \eta_r, \eta_{new})$, the last term of Eq 11 is a Dirichlet density:

$$p(\epsilon \mid \eta) = \text{Dir} \left( \frac{\eta_1, \eta_2, ..., \eta_M, \eta_{new}}{M} \right)$$

$$= \frac{\Gamma(M \times \eta_r + \eta_{new})}{\Gamma(\eta_r)^M \eta_{new}} \prod_{m=1}^{M} \epsilon_m^{\eta_r - 1} \epsilon_{new}^{-1}$$

Using the result:

$$\frac{\Gamma(v\epsilon_m + n_{km})}{\Gamma(v\epsilon_m)} = \sum_{a_{km} = 0}^{n_{km}} \text{Stirl}(a_{km}, n_{km}) (v\epsilon_m)^{a_{km}}$$

Thus, Eq 11 becomes:
The probability of the auxiliary variable $o_{km}$ is computed as:

$$p(o_{km}) = \sum_{o_{km}=0}^{n_{km}} \text{Stirl}(o_{km}, n_{km})(v\epsilon_m)^{o_{km}}$$

Now let $o = (o_{km} : \forall k, m)$ we derive the following joint distribution:

$$p(\epsilon | o, l, z, v, \eta) = \epsilon_{\text{new}}^{-1} \prod_{k=1}^{K} \Gamma(v) \prod_{m=1}^{M} \epsilon_{m}^{\eta_{m}-1} \sum_{o_{km}=0}^{n_{km}} \text{Stirl}(o_{km}, n_{km})(v\epsilon_m)^{o_{km}}$$

As $R \to \infty$, we have

$$p(\epsilon | o, l, z, v, \eta) \propto \epsilon_{\text{new}}^{-1} \prod_{m=1}^{M} \epsilon_{m}^{\sum_{K}^{o_{km}} - \eta_{m} - 1}$$

Finally, we sample $\epsilon$ jointly with the auxiliary variable $o_{km}$ by:

$$p(o_{km} = h | \cdot) \propto \text{Stirl}(h, n_{km})(v\epsilon_m)^{h}, \ h = 0, 1, \ldots, n_{km}$$

$$p(\epsilon) \propto \epsilon_{\text{new}}^{-1} \prod_{m=1}^{M} \epsilon_{m}^{\sum_{K}^{o_{km}} - 1}$$

**Sampling hyperparameters**

In the proposed model, there are three hyper-parameters which need to be sampled: $\alpha, v$ and $\eta$.

**Sampling $\eta$**

Using similar strategy and using technique from Escobar and West [3], we have

$$p(M | \eta, u) = \text{Stirl}(M, u) \eta^M \frac{\Gamma(\eta)}{\Gamma(\eta + u)}$$

where $u = \sum_{m} u_{m}$ with $u_{m} = \sum_{K}^{o_{km}}$ is in the previous sampling $\epsilon$ and $M$ is the number of active content atoms. Let $\eta \sim \text{Gamma}(\eta_1, \eta_2)$. Recall that:

$$\frac{\Gamma(\eta)}{\Gamma(\eta + u)} = \int_{0}^{1} t^{\eta}(1 - t)^{u-1} \left(1 + \frac{u}{\eta}\right) dt$$

that we have just introduced an auxiliary variable $t$
where

\( p(t \mid \eta) \propto t^\eta \eta^{-1} = \text{Beta}(\eta + 1, u) \)

Therefore,

\[
p(\eta \mid t) \propto \eta^{\eta_2-1+M} \exp \{-\eta \eta_2\} \times t^\eta \eta^{-1} \left(1 + \frac{u}{\eta}\right)
\]

\[
= \eta^{\eta_2-1+M} \times \exp \{-\eta(\eta_2 - \log t)\} \times (1-t)^{\eta-1} + \eta^{\eta_2-1+M} \exp \{-\eta(\eta_2 - \log t)\} \times (1-t)^{\eta-1} u
\]

\[
\propto \eta^{\eta_2-1+M} \exp \{-\eta(\eta_2 - \log t)\} + u\eta^{\eta_2-1+M-1} \exp \{-\eta(\eta_2 - \log t)\}
\]

\[
= \pi_t \Gamma(\eta_1 + M, \eta_2 - \log t) + (1 - \pi_t) \Gamma(\eta_1 + M - 1, \eta_2 - \log t) \tag{12}
\]

where \( \pi_t \) satisfies this following equation to make the above expression a proper mixture density:

\[
\frac{\pi_t}{1 - \pi_t} = \frac{\eta_1 + M - 1}{u(\eta_2 - \log t)} \tag{13}
\]

To re-sample \( \eta \), we first sample \( t \sim \text{Beta}(\eta + 1, u) \), compute \( \pi_t \) as in equation 13, and then use \( \pi_t \) to select the correct Gamma distribution to sample \( \eta \) as in Eq. 12.

**Sampling \( \alpha \)**

Again sampling \( \alpha \) is similar to Escobar et al [3]. Assuming \( \alpha \sim \Gamma(\alpha_1, \alpha_2) \) with the auxiliary variable \( t \):

\[
p(t \mid \alpha, K) \propto \alpha \Gamma(1 - t)^{J-1}
\]

\[
p(t \mid \alpha, K) \propto \text{Beta}(\alpha_1 + 1, J)
\]

\( J \): number of document

\[
p(\eta \mid t, K) \sim \pi_t \Gamma(\alpha_1 + K, \alpha_2 - \log(t)) + (1 - \pi_t) \Gamma(\alpha_1 + K - 1, \alpha_2 - \log(t))
\]

where \( c, d \) are prior parameter for sampling \( \eta \) following Gamma distribution and \( \frac{\pi_t}{1 - \pi_t} = \frac{\alpha_1 + K - 1}{J(\alpha_2 - \log t)} \)

**Sampling \( v \)**

Sampling \( v \) is similar to sampling concentration parameter in HDP [6]. Denote \( o_{km} = \sum_m o_{km} \), where \( o_{km} \) is defined previously during the sampling step for \( \epsilon \), \( n_{ks} = \sum_m n_{km} \), where \( n_{km} \) is the count of \( \{|l_{ji} \mid z_{ji} = k, l_{ji} = m\} \). Using similar technique in [6], we write:

\[
p(o_{1s}, o_{2s}, \ldots, o_{Ks} \mid v, n_{1s}, \ldots, n_{Ks}) = \prod_{k=1}^{K} \text{Stirl}(n_{ks}, o_{ks}) \alpha_{0s}^{o_{ks}} \frac{\Gamma(v)}{\Gamma(v + n_{ks})}
\]

where the last term can be expressed as

\[
\frac{\Gamma(v)}{\Gamma(v + n_{ks})} = \frac{1}{\Gamma(n_{ks})} \int_0^1 b_k^v (1 - b_k)^{n_{ks} - 1} \left(1 + \frac{n_{ks}}{v}\right) db_k
\]

Assuming \( v \sim \Gamma(v_1, v_2) \), define the auxiliary variables \( b = (b_k \mid k = 1, \ldots, K), b_k \in [0, 1] \) and \( t = (t_k \mid k = 1, \ldots, K), t_k \in \{0, 1\} \) we have

\[
q(v, b, t) \propto v^{n_{1s} + \sum_k M_k} \exp \{-v v_1\} \prod_{k=1}^{K} b_k^v (1 - b_k)^{M_k - 1} \left(\frac{M_k}{v}\right)^{t_k}
\]
We will sample the auxiliary variables $b_k, t_k$ in accordance with $v$ that are defined below:

\[
q(b_k \mid v) = \text{Beta}(v + 1, o_k^*)
\]

\[
q(t_k \mid .) = \text{Bernoulli} \left( \frac{o_k^*/v}{1 + o_k^*/v} \right)
\]

\[
q(v \mid .) = \text{Gamma} \left( v_1 + \sum_k (o_k^* - t_k), v_2 - \sum_k \log b_k \right)
\]

3 Relative Roles of Context and Content Data

Regarding the inference of the cluster index $z_j$ (Eq. 5), to obtain the marginal likelihood (the third term in Eq. 5) one has to integrate out the words’ topic labels $l_{ji}$. In doing so, it can be shown that the sufficient statistics coming from the content data toward the inference of the topic frequencies and the clustering labels will just be the empirical word frequency from each document. As each document becomes sufficiently long, the empirical word frequency quickly concentrates around its mean by the central limit theorem (CLT), so as soon as the effect of CLT kicks in, increasing document length further will do very little in improving this sufficient statistics.

Increasing the document length will probably not hurt, of course. But to what extent it contributes relative to the number of documents awaits a longer and richer story to be told.

We confirm this argument by varying the document length and the number of documents in the synthetic document and see how they affect the posterior of the clustering labels. Each experiment is repeated 20 times. We record the mean and standard deviation of the clustering performance by NMI score. As can be seen from Fig 2, using context observation makes the model more robust in recovering the true document clusters.

4 Perplexity Evaluation

The standard perplexity proposed by Blei et al [2], used to evaluate the proposed model as following:

\[
\text{perplexity} \left( w_{\text{Test}} \right) = \exp \left\{ - \frac{\sum_{j=1}^{J_{\text{Test}}} \log p \left( w_{\text{Test}j} \right)}{\sum_{j=1}^{J_{\text{Test}}} N_{\text{Test}j}} \right\}
\]

During individual sampling iteration $t$, we utilize the important sampling approach [5] to compute $p \left( w_{\text{Test}} \right)$. The posterior estimation of $\psi_m$ in a Multinomial-Dirichlet case is defined below, note that it can be in other types of conjugacies [4] (e.g. Gaussian-Wishart, Binomial-Poisson):

\[
\psi_{m,v}^t = \frac{n_{m,v}^t + \text{smooth}}{\sum_{v=1}^{V} n_{m,v}^t + V \times \text{smooth}}
\]

\[
\tau_{k,m}^t = \frac{c_{k,m} + vv \times \epsilon_m}{\sum_{m=1}^{M} (c_{k,m} + vv \times \epsilon_m)}
\]

where $n_{m,v}^t$ is number of times a word $v, v \in \{1, ..., V\}$ is assigned to context topic $\psi_m$ in iteration $t$, and $c_{k,m}$ is the count of the set $\{w_{ji} \mid z_j = k, l_{ji} = m, 0 \leq j \leq J, 0 \leq i \leq N_j \}$. There is a constant smooth parameter [1] that influence on the count, roughly set as 0.1. Supposed that we estimate $z_{\text{Test}j} = k$ and $l_{\text{Test}ji} = m$, then the probability $p \left( w_{\text{Test}j} \right)$ is computed as:

\[
p \left( w_{\text{Test}j} \right) = \prod_{i=1}^{N_{\text{Test}j}} \frac{1}{T} \sum_{t=1}^{T} \tau_{k,m}^t \psi_{m,w_{\text{Test}ji}}^t
\]

where $T$ is the number of collected Gibbs samples.
$J$: number of document.
$NJ$: number of word per document.
NMI: normalized mutual information.

MC2 on Synthetic Data

**Figure 2:** Document clustering performance with different numbers of observed words and documents.

Note: Document clustering performance is evaluated on the estimated document cluster $z_j$ vs their groundtruth.
References


