A MOREAU-YOSIDA APPROXIMATION SCHEME FOR HIGH-DIMENSIONAL POSTERIOR AND QUASI-POSTERIOR DISTRIBUTIONS

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ABSTRACT. Exact-sparsity inducing prior distributions in high-dimensional Bayesian analysis typically lead to posterior distributions that are very challenging to handle by standard Markov Chain Monte Carlo methods. We propose a methodology to derive a smooth approximation of such posterior distributions. The approximation is obtained from the forward-backward approximation of the Moreau-Yosida regularization of the negative log-density. We show that the derived approximation is within a factor $O(\sqrt{\gamma})$ of the true posterior distribution, where $\gamma > 0$ is a user-controlled parameter that defines the approximation. We illustrate the method with a variable selection problem in linear regression models, and with a Gaussian graphical model selection problem.

1. INTRODUCTION

Successful handling of statistical models with large number of parameters from limited amount of data hinges on the ability to solve efficiently and simultaneously two problems: (a) weeding out non-significant variables, and (b) estimating the effect of the significant variables. The concept of sparsity has come to play a fundamental role in this endeavor (Bühlmann and van de Geer (2011)). In the Bayesian framework, sparsity is naturally built in the prior distribution, and this can be done either approximately (weak sparsity), or exactly (exact sparsity). Weak sparsity is commonly implemented by employing continuous shrinkage prior distributions (Park and Casella (2008); Carvalho et al. (2010); Hans (2011); Armagan et al. (2013)). This approach leads to posterior distributions that are relatively easy to handle in practice, but has the drawback that it does not perform the variable selection step. Exact sparsity is
implemented using prior distributions that are mixtures of continuous and discrete distributions (George and McCulloch (1997); Ishwaran and Rao (2005); Bottolo and Richardson (2010); Narisetty and He (2014)). The recent work Castillo et al. (2014) suggests that these prior distributions can produce posterior distributions with optimal statistical properties. However, on the flip side, these posterior distributions are computationally very challenging to handle in practice.

The most commonly used approach for dealing with posterior distributions from exact-sparsity inducing priors is to exploit conjugacy and integrate out the regression coefficients (George and McCulloch (1997); Ishwaran and Rao (2005); Bottolo and Richardson (2010); Rockova and George (2014); Narisetty and He (2014)). However, this approach does not easily extend to non-conjugate prior distributions (for instance, for a spike-and-slab prior distribution for a Gaussian linear regression, conjugacy restricts the “slab” distribution to be a Gaussian distribution). Nor can it be easily extended to other types of models (generalized linear models, graphical models, etc.). Another solution to dealing with these posterior distributions is to design MCMC samplers using trans-dimensional MCMC techniques, including reversible jump MCMC (Chen et al. (2011)), or the newly proposed shrinkage-thresholding Metropolis adjusted Langevin algorithm (STMaLa; Schreck et al. (2013)). Alas, for high-dimensional models, sampling from this type of posterior distributions is an intrinsically hard problem as it requires jumping through an exponentially large number of sub-models.

The discussion above suggests that when dealing with sparsity inducing prior distributions in high-dimensional regression problems, good approximation of the posterior distribution would be useful. However, classical posterior approximation methods such as the Laplace approximation (Tierney and Kadane (1986)) cannot be straightforwardly applied. One of the issues with the Laplace approximation in this context is that it requires the log-density to be twice-differentiable at its mode, an assumption which typically fails to hold. Another issue is that the behavior of the Laplace approximation is poorly understood (and the method can fail) when the dimension of the space is as big as the sample size (Shun and McCullagh (1995)). Variational Bayes approximations form another popular class of approximation method that has been explored recently by Ormerod et al. (2014) for dealing with Bayesian variable selection with sparsity inducing prior distributions. These authors focus on the low-dimensional case where the number of regressors is smaller than the sample size and show that variational Bayes methods can approximate the posterior mean accurately. It is unclear whether the method can perform equally well in the high-dimensional case.
setting. It is also well-known that variational Bayes methods can poorly approximate other aspects of the posterior distribution, even in the low-dimensional setting.

1.1. Main contribution. We propose a general methodology to derive a smooth approximation of posterior distributions derived from exact-sparsity inducing prior distributions. The approximation removes the need for trans-dimensional MCMC. Furthermore, the fact that the approximation is smooth makes it possible to use well-established gradient-based methods for efficient MCMC simulation. The approximation is obtained by taking the forward-backward approximation of the Moreau-Yosida approximation of the negative log-density. The Moreau-Yosida approximation is a well-established regularization method in optimization for dealing with non-smooth and constrained problems (Moreau (1965); Brézis (1973); Martinet (1972); Rockafellar (1976); Bauschke and Combettes (2011)). Several recent works have recognized the usefulness of the Moreau-Yosida regularization for Bayesian computation. Pereyra (2013) noted that a log-concave density can be well approximated by its Moreau-Yosida approximation. However this paper does not study the quality of the approximation, and focuses on using the smoothness properties of the Moreau-Yosida approximation to design MCMC algorithms to sample from the target distribution itself. Another related work is the STMaLa of Schreck et al. (2013) mentioned above, which implicitly uses the smoothness of the Moreau approximation to design Metropolis-Hastings proposals to sample from posterior distributions with exact-sparsity inducing prior distributions.

To broaden the applicability of the paper, we work in the more general framework of quasi-posterior distributions (Chernozhukov and Hong (2003)). Our methodology hence produces approximate quasi-posterior distributions. We derive a general result that shows, under some regularity conditions that the proposed approximate quasi-posterior distribution is within a factor $O(\sqrt{\gamma})$ of the true quasi-posterior distribution, where $\gamma > 0$ is a user-controlled parameter that defines the quality of the approximation (see Theorem 3). The key condition that we impose to guarantee the correctness of the approximation, is that the quasi-posterior distribution is log-concave.

We illustrate the method using a linear regression model, and a Gaussian graphical model. We use the linear regression example to show by simulation that the proposed method is order of magnitude faster than STMaLa. In the linear regression setting, our approach bears some similarity with the approach to Bayesian variable selection proposed by George and McCulloch (1997); Rockova and George (2014), where the spike point-mass at zero is replaced by another Gaussian distribution with very small variance. Using the Moreau-Yosida approximation achieves a similar effect, with the
additional feature that the resulting distribution is smooth in the regression parameters (even if the “slab” density is not differentiable at zero). In the second illustration, we use the proposed methodology to sample from a quasi-posterior distribution for a Gaussian graphical model selection problem. This example illustrates the usefulness of the quasi-Bayesian framework, and illustrates our point that the proposed approximation method can be applied very broadly.

The remainder of the paper is organized as follows. We close the introduction with some notation that will be used throughout the paper. We present a general quasi-Bayesian framework using exact-sparsity inducing prior distributions in Section 2, followed in Section 3 by its Moreau-Yosida approximation. Section 4 and Section 5 detail the applications to linear regression models, and Gaussian graphical models respectively. We close the paper with further discussion in Section 6. All the proofs are gathered in Section 7.

1.2. Notation. Throughout the paper the data space \( X^{(n)} \) is a measurable space equipped with a reference sigma-finite measure denoted \( d_x \). The upper script \( n \) represents the sample size and is explicitly written for improved readability (since \( n \) is kept fixed throughout). The parameter space \( \Theta \) is a non-empty convex open subset of the Euclidean space \( \mathbb{R}^d \), \( d \geq 1 \), equipped with its Borel sigma-algebra, its Euclidean norm \( \| \cdot \| \), and inner product \( \langle \cdot, \cdot \rangle \). We also use the norms \( \| \theta \|_1 \overset{\text{def}}{=} \sum_{j=1}^d |\theta_j| \), and \( \| \theta \|_\infty \overset{\text{def}}{=} \max_{1 \leq j \leq d} |\theta_j| \).

We set \( \Delta \overset{\text{def}}{=} \{0,1\}^d \). We write \( \mu_d \) for the (product) Lebesgue measure on \( \mathbb{R}^d \). For \( \delta \in \Delta \), \( \mu_{d,\delta} \) denote the product measure on \( \mathbb{R}^d \) defined as \( \mu_{d,\delta}(d\theta) = \prod_{j=1}^d \nu_{\delta_j}(d\theta_j) \), where \( \nu_0(dz) \) is the Dirac mass at 0, and \( \nu_1(dz) \) is the Lebesgue measure on \( \mathbb{R} \). We define \( \Theta_\delta \overset{\text{def}}{=} \{ \theta \in \Theta : \theta_j = 0, \text{ whenever } \delta_j = 0 \} \). For \( \theta, \theta' \in \Theta \), we denote \( \theta \cdot \theta' \) their component-wise product. Finally, in what follows, \( \Xi \) denotes an auxiliary parameter space equipped with a metric \( d \) and its associated Borel sigma-algebra.

2. Quasi-posterior distributions with sparse priors

Let \( x_n \in X^{(n)} \) be a realization of a random variable \( X_n \) with conditional distribution \( P_{\theta,\phi}(dx) = f_{\theta,\phi}(x)dx \) given a parameter \( (\theta, \phi) \in \Theta \times \Xi \). The parameter \( \theta \) is the main parameter of interest, and \( \phi \) is an auxiliary parameter. Let \( \Pi \) be the prior distribution of \( (\theta, \phi) \). The posterior distribution of \( (\theta, \phi) \) given the observed data \( x_n \) is

\[
\Pi_n(d\theta, d\phi | x_n) = \frac{f_{\theta,\phi}(x_n)\Pi(d\phi, d\theta)}{\int f_{\theta,\phi}(x_n)\Pi(d\phi, d\theta)}.
\]

In some applications it is desirable to replace the likelihood function \( f_{\theta,\phi} \) by a quasi-likelihood function. Hence, we consider a more general quasi-density function \( q_{\theta,\phi} : \)
$X^{(n)} \to [0, \infty)$, such that $0 < \int_{\Theta \times \Xi} q_{\theta, \phi}(x)\Pi(d\phi, d\theta) < \infty$, for all $x \in X^{(n)}$. Substituting $q_{\theta, \phi}$ in place of $f_{\theta, \phi}$ yields the quasi-posterior distribution

$$\tilde{\Pi}_n(d\theta, d\phi|x_n) = \frac{q_{\theta, \phi}(x_n)\Pi(d\phi, d\theta)}{\int q_{\theta, \phi}(x_n)\Pi(d\phi, d\theta)}.$$  

To the best of our knowledge, the use of quasi-posterior distributions is due to Chernozhukov and Hong (2003). There are various motivations for using these quasi-posterior distributions. In Chernozhukov and Hong (2003), the quasi-posterior framework was introduced as a more convenient alternative to optimizing a highly multi-modal objective function. Yang and He (2012) employed a quasi-posterior framework to deal with a semi-parametric quantile regression model in a Bayesian setting. We detail in Section 5 an example with Gaussian graphical models, where the quasi-posterior framework helps mitigate some of the computational issues related to the intractable normalizing constants. From the viewpoint of statistical inference, some contraction properties of quasi-posterior distributions can be found in Kato (2013), but much more remains to be done in that direction.

Although the prior distribution $\Pi$ can be constructed in a variety of ways, we focus on exact-sparsity inducing prior distributions for $\theta$. More specifically, we consider a prior distribution $\Pi$ on $\Delta \times \Xi \times \Theta$ of the form

$$\Pi(\delta, d\phi, d\theta) = \pi_\delta G(d\phi)\Pi(d\theta|\delta, \phi),$$

for a discrete distribution $\{\pi_\delta, \delta \in \Delta\}$ on $\Delta$, a prior probability measure $G$ on $\Xi$, and a sparsity inducing prior $\Pi(\cdot|\delta, \phi)$ on $\theta$. We build the prior on $\theta$ as follows. Given $\delta, \phi$, the components of $\theta$ are independent, and for $1 \leq j \leq d$,

$$\theta_j|\delta, \phi \sim \begin{cases} 
\text{Dirac}(0) & \text{if } \delta_j = 0 \\
p(\cdot|\phi) & \text{if } \delta_j = 1
\end{cases},$$

(1)

where $\text{Dirac}(0)$ is the Dirac measure on $\mathbb{R}$ with full mass at 0, and $p(\cdot|\phi)$ is a positive density on $\mathbb{R}$ with respect to the Lebesgue measure. With the measure $\mu_{d, \delta}$ defined in Section 1.2 this prior can be written as

$$\Pi(d\theta|\delta, \phi) = \exp \left( \sum_{j=1}^{d} \delta_j \log p(\theta_j|\phi) \right) \mu_{d, \delta}(d\theta).$$

We obtain the quasi-posterior distribution $\tilde{\Pi}_n$ on $\Delta \times \Xi \times \Theta$ given by

$$\tilde{\Pi}_n(\delta, d\phi, d\theta|x_n) \propto \pi_\delta q_{\theta, \phi}(x_n) \exp \left( \sum_{j=1}^{d} \delta_j \log p(\theta_j|\phi) \right) G(d\phi)\mu_{d, \delta}(d\theta).$$

(2)

When $x \mapsto q_{\theta, \phi}(x)$ is not a density, it is not automatic that (2) is a well-defined probability measure on $\Delta \times \Xi \times \Theta$. This needs to be checked separately, similarly to when
using improper priors. However, even if $\hat{\Pi}_n$ is well-defined, simulating from this distribution is challenging. The issue is related to the discrete-continuous mixture form of the prior distribution on $\theta$, which has the effect that any two conditional distributions $\hat{\Pi}_n(\cdot|\theta, \delta)$ and $\hat{\Pi}_n(\cdot|\theta, \delta')$ are mutually singular for $\delta \neq \delta'$. As a result, sampling from the distribution (2) requires the use of trans-dimensional MCMC methods such as reversible jump (Chen et al. (2011)), or the recently proposed STMaLa (Schreck et al. (2013)). However, and precisely because of the trans-dimensional nature of the distribution, these samplers are required to jump from sub-space to sub-space in a discrete fashion (although the STMaLa has a more savvy approach to doing this). When the dimension $d$ is large, the mixing of these algorithms can be problematic.

We propose a smooth approximation of (2) that is easier to explore by standard MCMC. The approximation is based on the forward-backward approximation of the Moreau-Yosida approximation well-known in convex optimization.

3. The Moreau-Yosida quasi-posterior approximation

We introduce the following functions: for $(\delta, \phi, \theta) \in \Delta \times \Xi \times \Theta$,

$$\ell_n(\theta|\phi) \overset{\text{def}}{=} - \log q_{\theta,\phi}(x_n), \quad P_n(\theta|\delta, \phi) \overset{\text{def}}{=} - \sum_{j=1}^{d} \delta_j \log p(\theta_j|\phi) + \iota_{\Theta_\delta}(\theta), \quad (3)$$

where $\iota_{\Theta_\delta}(\theta) = 0$ if $\theta \in \Theta_\delta$, and $\iota_{\Theta_\delta}(\theta) = +\infty$ otherwise. For $(\delta, \phi) \in \Delta \times \Xi$, we define

$$h_n(\theta|\delta, \phi) \overset{\text{def}}{=} \ell_n(\theta|\phi) + P_n(\theta|\delta, \phi), \quad \theta \in \Theta.$$

Throughout the paper we impose the following assumption.

**H1.** (1) For all $\phi \in \Xi$, the function $\theta \mapsto \ell_n(\theta|\phi)$ introduced in (3) is convex and differentiable, with a gradient that is Lipschitz with Lipschitz constant $L \in (0, \infty)$. Furthermore, the Lipschitz constant $L$ does not depend on $\phi$.

(2) For all $(\delta, \phi) \in \Delta \times \Xi$, the function $\theta \mapsto P_n(\theta|\delta, \phi)$ is convex.

**Remark 1.** (1) The convexity assumption on $\ell_n$ is fundamental and delineates the type of problems to which the proposed approximation could be easily applied. Extension beyond this set up is possible, but will require fundamentally different techniques.

(2) The assumption that the Lipschitz constant $L$ does not depend on $\phi$ is restrictive. If the Lipschitz is $L_\phi$ and depends on $\phi$, as naive approach to recover $H1$ is to assume that $\phi$ belong to a compact set $\Xi$, and take $L = \sup_{\phi \in \Xi} L_\phi$. Unfortunately, this easy fix typically does not result in a good Moreau-Yosida approximation. It is currently not clear to us how to effectively extend the methodology to allow $L$ to depend on $\phi$. 

(3) Notice that the sets \( \Theta_\delta \) are convex nonempty sets, hence the convexity of 
\( P_n(\cdot|\delta, \phi) \) boils down to the log-concavity of the density \( p \) in the prior \( \mathcal{I} \).

Most of the sparsity promoting prior densities used in practice are log-concave.

Using the function \( h_n \), the mutual singularity of the measures \( \Pi_n(\delta, \cdot|x_n) \) manifests itself through the fact that \( h_n(\theta|\delta, \phi) = +\infty \) for \( \theta \not\in \Theta_\delta \), and \( \Theta_\delta \cap \Theta_{\delta'} = \{0\} \) for \( \delta \neq \delta' \). To deal with this issue we propose to replace \( h_n(\cdot|\delta, \phi) \) by the forward-backward approximation of its Moreau-Yosida approximation \(^2\) defined as

\[
h_{n, \gamma}(\theta|\delta, \phi) \equiv \min_{u \in \Theta} \left[ \ell_n(\theta|\phi) + \langle \nabla \ell_n(\theta|\phi), u - \theta \rangle + P_n(u|\delta, \phi) + \frac{1}{2\gamma} \| u - \theta \|^2 \right],
\]

for some parameter \( \gamma > 0 \). In the above equation, \( \nabla \ell_n(\theta|\phi) \) denotes the gradient of \( u \mapsto \ell_n(u|\phi) \) evaluated at \( \theta \). Basic properties of \( h_{n, \gamma} \) can be found in Patrinos et al. (2014), which uses the term forward-backward approximation of \( h_n \). The function \( h_{n, \gamma}(\cdot|\delta, \phi) \) can be evaluated via the Moreau-Yosida approximation of \( P_n(\cdot|\delta, \phi) \). More specifically, for \( \delta \in \Delta, \phi \in \Xi \), and for \( \gamma > 0 \), we define

\[
P_{n, \gamma}(\theta|\delta, \phi) \equiv \min_{u \in \Theta} \left[ P_n(u|\delta, \phi) + \frac{1}{2\gamma} \| u - \theta \|^2 \right],
\]

known as the Moreau-Yosida regularization of \( P_n \), and its associated proximal map

\[
\text{Prox}_{\gamma}^P(\theta|\delta, \phi) \equiv \text{Argmin}_{u \in \Theta} \left[ P_n(u|\delta, \phi) + \frac{1}{2\gamma} \| u - \theta \|^2 \right],
\]

Notice that \( \text{Prox}_{\gamma}^P(\theta|\delta, \phi) \in \Theta_\delta \). This is because by definition (see (3)), \( P_n(\theta|\delta, \phi) = +\infty \) for \( \theta \not\in \Theta_\delta \), and \( \Theta_\delta \) is non-empty. From the definition of \( P_{n, \gamma} \) and \( \text{Prox}_{\gamma}^P \), we see that \( h_{n, \gamma} \) can be alternatively written as

\[
h_{n, \gamma}(\theta|\delta, \phi) = \ell_n(\theta|\phi) - \frac{\gamma}{2} \| \nabla \ell_n(\theta|\phi) \|^2 + P_{n, \gamma}(\theta - \gamma \nabla \ell_n(\theta|\phi)|\delta, \phi)
\]

\[
= \ell_n(\theta|\phi) + \langle \nabla \ell_n(\theta|\phi), J_{\gamma}(\theta|\delta, \phi) - \theta \rangle + P_{n}(J_{\gamma}(\theta|\delta, \phi)|\delta, \phi)
\]

\[
+ \frac{1}{2\gamma} \| J_{\gamma}(\theta|\delta, \phi) - \theta \|^2,
\]

where

\[
J_{\gamma}(\theta|\delta, \phi) \equiv \text{Prox}_{\gamma}^P(\theta - \gamma \nabla \ell_n(\theta|\phi)|\delta, \phi).
\]

The function \( \theta \mapsto h_{n, \gamma}(\theta|\delta, \phi) \) has many nice properties. Under \( \mathcal{H} \), it is convex, continuous differential (Patrinos et al. (2014) Theorem 2.2), satisfies \( h_{n, \gamma}(\theta|\delta, \phi) \leq h_n(\theta|\delta, \phi) \), and converges to \( h_n(\theta|\delta, \phi) \) as \( \gamma \to 0 \) (see Lemma \( \mathcal{I} \) below). Besides the fact that \( h_{n, \gamma}(\theta|\delta, \phi) \) approximates \( h_n(\theta|\delta, \phi) \), what makes \( h_{n, \gamma} \) particularly interesting in

\(^2\)The Moreau-Yosida approximation of the function \( h_n \) itself is the function \( \theta \mapsto \min_{u \in \Theta} \left[ h_n(u|\delta, \phi) + \frac{1}{2\gamma} \| u - \theta \|^2 \right] \). This function is typically intractable. \( \mathcal{I} \) is an approximation obtained by replacing \( \ell_n(u|\phi) \) by its linear approximation \( \ell_n(\theta|\phi) + \langle \nabla \ell_n(\theta|\phi), u - \theta \rangle \).
the present context is the fact that $h_{n, \gamma}(\theta|\delta, \phi)$ is finite for all $\theta \in \Theta$ (unlike $h_n(\cdot|\delta, \phi)$ which is $+\infty$ for $\theta \notin \Theta_\delta$). Using $h_{n, \gamma}$, we now approximate the quasi-posterior distribution $\hat{\Pi}_n$ in (2) by

$$\hat{\Pi}_{n, \gamma}(\delta, d\phi, d\theta|x_n) \propto \pi_{\delta} \left(2\pi \gamma \right)^{\|\delta\|^2/2} e^{-h_{n, \gamma}(\theta|\delta, \phi)} G(d\phi) \mu_d(d\theta), \quad (8)$$

that we call the Moreau-Yosida approximation of $\hat{\Pi}_n$, although strictly speaking, (4) is only the forward-backward approximation of the Moreau-Yosida approximation of $h_n(\cdot|\delta, \phi)$. In this expression (8), $\pi$ represents the irrational number. The factor $(2 \pi \gamma)^{\|\delta\|^2/2}$ appearing in (8) accounts for the fact that the Moreau-Yosida approximation perturbs the marginal distribution of $\delta$ by the multiplicative factor $(2 \pi \gamma)^{d-\|\delta\|^2/2}$ which needs to be corrected. The approximation $\hat{\Pi}_{n, \gamma}$ has the effect of decoupling the variables $\delta_j$ and $\theta_j$ (for any given $j$), in the sense that all the $\theta_j$ now have a continuous density, and $\delta_j = 0$ no longer imply that $\theta_j = 0$. Typically, when $\delta_j = 0$, the distribution of $\theta_j$ has a density centered around 0 with a very small variance. This decoupling has little practical consequence, since one can still infer from the marginal distribution of $\delta_j$ whether $\delta_j = 0$ or not.

One can think of (8) as the distribution obtained by replacing all the point-mass priors in the quasi-posterior distribution $\hat{\Pi}_n$ in (2) by independent copies of the density of the Gaussian distribution $N(0, \gamma)$. However, owing to the Moreau-Yosida approximation, this is done in such a way that the resulting function $h_{n, \gamma}(\theta|\delta, \phi)$ is smooth in $\theta$ (even if the prior density $p$ used in (1) is not smooth). It is then possible to take advantage of this smoothness property to explore $\hat{\Pi}_{n, \gamma}$ by MCMC, more efficiently than we would be able to do for the quasi-posterior distribution $\hat{\Pi}_n$ itself. Furthermore, as $\gamma \rightarrow 0$, we will see that $\hat{\Pi}_{n, \gamma}$ converges to $\hat{\Pi}_n$. Since $\hat{\Pi}_n$ is a difficult distribution to sample from, naturally, sampling from $\hat{\Pi}_{n, \gamma}$ becomes progressively hard, as $\gamma \rightarrow 0$. Hence in order to use (8), a good understanding of this trade-off (between good approximation properties and efficient sampling) is needed in order to make sensible choice of $\gamma$. A deeper investigation of this question is beyond the scope of this work, but we provide some guideline for choosing $\gamma$ in the examples below.

We will now derive a result that shows that $\hat{\Pi}_{n, \gamma}$ is a well-defined probability measure that converges to $\hat{\Pi}_n$, as $\gamma \rightarrow 0$. We make the following assumption.

**H2.** For all $(\delta, \phi) \in \Delta \times \Xi$, there exists a measurable function $g_n(\cdot|\delta, \phi) : \Theta \rightarrow \mathbb{R}^d$, such that for all $\theta \in \Theta_\delta$, $g_n(\theta|\delta, \phi)$ belongs to the sub-differential of the function $P_n(\cdot|\delta, \phi)$ at $\theta$. Furthermore, there exists $\gamma_0 > 0$, such that

$$\max_{\delta \in \Delta} \int_{\Xi} \left[ \int_{\Theta} e^{\gamma_0 r_n(\theta|\delta, \phi)} e^{-h_n(\theta|\delta, \phi)} \mu_d(d\theta) \right] G(d\phi) < \infty,$$

where $r_n(\theta|\delta, \phi) \overset{\text{def}}{=} \|g_n(\theta|\delta, \phi)\|^2 + \|\nabla \ell_n(\theta|\phi)\|^2$. 


Remark 2. Notice that $H_2$ implies that the quasi-posterior distribution $\Pi_n$ itself is well-defined. In many cases, $r_n(\theta|\delta, \phi)$ and $\ell_n(\theta|\phi)$ behave similarly as $\theta \to \infty$. For such cases, $H_2$ can be shown to hold, as we will see in the examples below.

Under this additional assumption we will show that $\Pi_{n, \gamma}$ converges weakly to $\Pi_n$ as $\gamma \to 0$. For convenience we introduce the product space $\Theta_1 \times \Xi \times \Theta$. For any two probability measures $\nu_1, \nu_2$ on $\Theta$, the $\beta$-distance between $\nu_1, \nu_2$ is defined as

$$\beta(\nu_1, \nu_2) = \sup_{\|f\|_B \leq 1} \left| \int_\Theta f(\theta) \nu_1(d\theta) - \int_\Theta f(\theta) \nu_2(d\theta) \right|,$$

where the supremum is taken over all measurable functions $f : \Theta \to \mathbb{R}$ such that

$$\|f\|_B = \|f\|_\infty + \sup \left\{ \frac{|f(\theta_1) - f(\theta_2)|}{d(\theta_1, \theta_2)} : \theta_1, \theta_2 \in \Theta, \theta_1 \neq \theta_2 \right\} \leq 1,$$

and $d(\theta_1, \theta_2) = \sqrt{\|\delta_1 - \delta_2\|^2 + d(\phi_1, \phi_2)^2 + \|\theta_1 - \theta_2\|^2}$. It is well known that the metric $\beta$ metricizes weak convergence (see Dudley (2002) Theorem 11.3.3). For the need of the theorem, we introduce

$$\varphi_\gamma = \max_{\delta \in \Delta} \frac{\int_\Xi \gamma^{-1} \left( e^{\gamma r_n(\theta|\delta, \phi)} - 1 \right) e^{-h_n(\theta|\delta, \phi)} \mu(d\theta)}{\int_\Xi e^{-h_n(\theta|\delta, \phi)} \mu(d\theta)} G(d\phi), \quad \gamma > 0.$$

Theorem 3. Assume $H_1, H_2$. Take $\gamma_0 > 0$ as in $H_2$, and such that $4\gamma_0 L \leq 1$, where $L$ is as in $H_1$. Then for all $\gamma \in (0, \gamma_0)$, $\Pi_{n, \gamma}(\cdot|x_n)$ is a well-defined probability measure on $\Theta$. Furthermore, if $d \geq 2$, then for all $\gamma \in (0, \gamma_0)$,

$$\beta(\Pi_{n, \gamma}, \Pi_n) \leq \sqrt{\gamma d} + 2\gamma \left( \varphi_\gamma + \frac{5}{2} Ld \right). \quad (9)$$

Proof. See Section 7.1. $\square$

Remark 4. Since $e^x \leq 1 + xe^x$ for all $x \geq 0$, we have

$$\gamma^{-1} \left( e^{\gamma r_n(\theta|\delta, \phi)} - 1 \right) \leq r_n(\theta|\delta, \phi) e^{\gamma r_n(\theta|\delta, \phi)}.$$

Hence, if

$$\max_{\delta \in \Delta} \int_\Xi \left[ \int_\Theta r_n(\theta|\delta, \phi) e^{\gamma r_n(\theta|\delta, \phi)} e^{-h_n(\theta|\delta, \phi)} \mu(d\theta) \right] G(d\phi) < \infty, \quad (10)$$

for some $\gamma_0 > 0$, (note that (10) is only slightly stronger than $H_2$), then $\varphi_\gamma$ is bounded in $\gamma$, and in this case the convergence rate in (9) is now seen to be $O(\sqrt{\gamma})$. One question that we were not able to resolve is understanding the dependence of $\varphi_\gamma$ as a function of the dimension $d$. This question would provide some insight on whether the method suffer from the curse of dimensionality. We conjecture that under (10), $\varphi_\gamma \leq C_0 d$, for $\gamma \in (0, \gamma_0)$, and for some universal constant $C_0$. 
4. Application to Bayesian linear regression with sparse priors

We consider the linear model

\[ y = X\theta + \epsilon, \]

where \( X \in \mathbb{R}^{n \times d} \) is a known design matrix, \( \theta \in \mathbb{R}^d \), and \( \epsilon \sim N(0, \sigma^2 I_n) \), for some noise parameter \( \sigma^2 > 0 \) that we assume known for the time being (see Section 4.3). An extensive literature has been developed in recent years to deal with linear regression models in the scenario where \( d \) is larger than \( n \) (Bühlmann and van de Geer (2011)). We focus here on the Bayesian formulation using exact-sparsity inducing priors. We will show that the methodology presented above provides a generally applicable strategy to approximate the resulting posterior distribution, and we will develop a MCMC algorithm to draw samplers from this approximation.

Since \( \sigma^2 \) is assumed known, the negative-log-likelihood function \( \ell_n \) for this problem can be taken as

\[ \ell_n(\theta | \sigma^2) = \frac{1}{2\sigma^2} \| y - X\theta \|^2. \]

We will set up the prior distribution of \( \theta \) using \( \delta \in \Delta \), and using an auxiliary variable \( \phi = (q, \lambda_1, \lambda_2) \), where \( q \in (0, 1) \) is a sparsity parameter, and \( \lambda_1 > 0, \lambda_2 > 0 \) are regularization parameters. Given \( \delta \in \Delta \), \( q \in (0, 1) \), and \( \lambda_1 > 0, \lambda_2 > 0 \), we assume that the components of \( \theta \) are independent, and for \( 1 \leq j \leq d \),

\[ \theta_j | \delta, \alpha, \lambda_1, \lambda_2 \sim \begin{cases} \text{Dirac}(0) & \text{if } \delta_j = 0 \\ p(\cdot | \alpha, \lambda_1, \lambda_2) & \text{if } \delta_j = 1 \end{cases}, \]

where \( \text{Dirac}(0) \) is the Dirac measure on \( \mathbb{R} \) with full mass at 0, and \( p(\cdot | \alpha, \lambda_1, \lambda_2) \) is the density on \( \mathbb{R} \) given by

\[ p(x | \alpha, \lambda_1, \lambda_2) = \frac{1}{Z} \exp \left( -\alpha \lambda_1 |x| - \frac{(1-\alpha)\lambda_2}{2} x^2 \right), \quad x \in \mathbb{R}. \quad (11) \]

The prior density (11) is a slight modification of the elastic-net (Zou and Hastie (2005)) prior used by Li and Lin (2010); Hans (2011). The parameter \( \alpha \in [0, 1] \) is assumed known throughout. Notice that setting \( \alpha = 1 \) makes \( \lambda_2 \) inactive, and setting \( \alpha = 0 \) makes \( \lambda_1 \) inactive. Beside these two extremes, the specific choice of \( \alpha \in (0, 1) \) is less important, because we will allow \( \lambda_1 \) and \( \lambda_2 \) to vary. In the simulations below we take \( \alpha = 0.9 \).

Other prior densities can be used provided that they are log-concave, and their associated proximal operators are easy to compute. The normalizing constant of the density \( p \) can be written as \( Z = Z_\alpha(\lambda_1, \lambda_2) \), where

\[ Z_\alpha(\lambda_1, \lambda_2) \overset{\text{def}}{=} \begin{cases} \sqrt{\frac{2\pi}{(1-\alpha)\lambda_2}} \text{erfcx} \left( \frac{\alpha \lambda_1}{\sqrt{2(1-\alpha)\lambda_2}} \right) & \text{if } \alpha \in [0, 1) \\ \frac{2}{\lambda_1} & \text{if } \alpha = 1 \end{cases}, \quad (12) \]
where \( \text{erfcx}(x) \) is the scaled complementary error function, which can be written as \( \text{erfcx}(x) = 2e^{-x^2}\Phi(-\sqrt{2}x) \), where \( \Phi \) is the cdf of standard normal distribution.

We assume that the sparsity parameter \( q \) is a uniform random variable \( q \sim U(0, 1) \), and given \( q \), the components of \( \delta \) are independent with a Bernoulli distribution \( \text{Ber}(q) \). Finally, we assume that \( \lambda_1 \) and \( \lambda_2 \) are uniformly distributed on \( U(a, M) \), for \( 0 < a < M < \infty \). The justification for this latter prior distribution is mostly theoretical: in Theorem 3 we need \( \lambda_1 \) and \( \lambda_2 \) to remain bounded and bounded away from zero. In the simulations below, we set \( a = 10^{-5} \), and we set \( M \) as the upper bound on \( M \) provided in Theorem 3. In summary, mapping this problem in the notation of Section 2, we have \( X^{(n)} \in \mathbb{R}^n \), \( \Theta = \mathbb{R}^d \), \( \phi = (q, \lambda_1, \lambda_2) \) and belongs to the set \( \Xi = (0, 1) \times (a, M) \times (a, M) \), and \( G \) is the uniform measure on \( \Xi \). Given the prior density \( p \) in (11), the function \( P_n(.|\delta, \phi) \) in (3) becomes

\[
P_n(\theta|\sigma^2, \delta, \phi) = |\delta|_1 \log Z + \frac{\alpha \lambda_1}{\sigma^2} |\theta|_1 \| \delta \|_1 + \frac{(1 - \alpha) \lambda_2}{2 \sigma^2} |\theta \cdot \delta|_2^2 + \iota \Theta(\theta),
\]

and with \( h_n(\theta|\sigma^2, \delta, \phi) = \ell_n(\theta|\sigma^2) + P_n(\theta|\sigma^2, \delta, \phi) \), the posterior distribution of \( (\delta, \theta, \phi) \) is

\[
\Pi_n(\delta, d\theta, d\phi|y, \sigma^2) \propto q^{|\delta|_1} (1 - q)^{d - |\delta|_1} e^{-h_n(\theta|\sigma^2, \delta, \phi)} \, d\phi \mu_{d, \delta}(d\theta).
\]

With the elastic net prior (11), the proximal function \( \text{Prox}_{\gamma}^{P_n}(\theta|\sigma^2, \delta, \phi) \) is easy to compute. More precisely, for \( \gamma > 0 \), let \( s_\gamma(\theta; \lambda_1, \lambda_2) \in \mathbb{R}^d \) denotes the vector whose \( j \)-th entry is given by

\[
(s_\gamma(\theta; \lambda_1, \lambda_2))_j = \frac{\text{sign}(\theta_j) (|\theta_j| - \alpha \gamma \lambda_1)_+}{1 + \gamma \lambda_2 (1 - \alpha)}.
\]

It is easy to show that

\[
\text{Prox}_{\gamma}^{P_n}(\theta|\sigma^2, \delta, \phi) = \delta \cdot s_\gamma \left( \theta; \frac{\lambda_1}{\sigma^2}, \frac{\lambda_2}{\sigma^2} \right),
\]

where we recall from (Section 1.2) that for \( \theta_1, \theta_2 \in \Theta \), \( \theta_1 \cdot \theta_2 \) denotes their component-wise product. From (5), it follows that the Moreau-Yosida approximation \( \Pi_{n, \gamma} \) of \( \Pi_n \) has a density \( \tilde{\pi}_{n, \gamma} \) given by

\[
\tilde{\pi}_{n, \gamma}(\delta, \theta, \phi|y, \sigma^2) \propto q^{|\delta|_1} (1 - q)^{d - |\delta|_1} (2\pi \gamma)^{|\delta|_1/2} \times e^{-h_{n, \gamma}(\theta|\delta, \sigma^2, \phi)},
\]

where (7) gives

\[
h_{n, \gamma}(\theta|\delta, \sigma^2, \phi) = \ell_n(\theta|\sigma^2) + \langle \nabla \ell_n(\theta|\sigma^2), J_{\gamma} - \theta \rangle
\]

\[
+ |\delta|_1 \log Z + \frac{\alpha \lambda_1}{\sigma^2} |J_{\gamma}|_1 + \frac{(1 - \alpha) \lambda_2}{2 \sigma^2} |J_{\gamma}|_2^2 + \frac{1}{2\gamma} |J_{\gamma} - \theta|^2,
\]

and

\[
J_{\gamma} = J_{\gamma}(\theta|\delta) = \delta \cdot s_\gamma \left( \theta - \gamma \nabla \ell_n(\theta|\sigma^2); \frac{\lambda_1}{\sigma^2}, \frac{\lambda_2}{\sigma^2} \right).
\]
In the next result, we show that H1 and H2 hold for this problem, and Theorem 3 applies. In the process, we provide some guideline for choosing $\gamma$. For a matrix $A$, let $\lambda_{\text{max}}(A)$ denote its largest eigenvalue.

**Theorem 5.** Take $\gamma_0 = \frac{\sigma^2}{4\lambda_{\text{max}}(X'X)}$, and suppose that $M < \frac{\lambda_{\text{max}}(X'X)}{(1-\alpha)}$. Then for all $\gamma \in (0, \gamma_0)$, $\tilde{\Pi}_{n,\gamma}$ is a well-defined probability measure on $\Delta \times \mathbb{R}^d \times \Xi$. Furthermore, if $d \geq 2$, there exists a finite constant $C$ (that depends on $d$) such that for all $\gamma \in (0, \gamma_0)$

$$\beta(\tilde{\Pi}_{n,\gamma}, \tilde{\Pi}_n) \leq \sqrt{\gamma d} + \gamma (C + 5Ld).$$

**Proof.** See Section 7.2 \hfill $\Box$

The result suggests choosing

$$\gamma = \frac{\gamma_0 \sigma^2}{\lambda_{\text{max}}(X'X)};$$  

with $\gamma_0 \in (0, 1/4]$. The simulation study conducted below suggest that values of $\gamma_0$ between 0.25 and 0.1 give good approximation of $\tilde{\Pi}_n$ and lead to a distribution $\tilde{\Pi}_{n,\gamma}$ that is easily explored by MCMC.

4.1. **Markov Chain Monte Carlo.** The density $\hat{\pi}_{n,\gamma}$ is a “standard” density, and various MCMC schemes can be used to sample from it. Here, we propose a Metropolized-Gibbs strategy, where we update $\delta$ keeping $(\theta, \phi)$ fixed, then we update $\theta$ keeping $(\delta, \phi)$ fixed, and we update $\phi = (\pi, \lambda_1, \lambda_2)$ keeping $(\delta, \theta)$ fixed.

4.1.1. **Updating $\delta$.** Given $\theta$ and $\phi$, it is easy to see that $h_{n,\gamma}(\theta|\sigma^2, \delta, \phi)$ depends on $\delta_j$ only through the expression

$$\delta_j \left[ (\nabla \ell_n(\theta|\sigma^2))_j d_j + \log Z + \frac{\alpha \lambda_1 |d_j| + 0.5(1-\alpha) \lambda_2 d_j^2}{\sigma^2} + \frac{d_j^2 - 2\theta_j d_j}{2\gamma} \right],$$

where $d_j$ is the $j$-th component of $s_{\gamma}(\theta - \gamma \nabla \ell_n(\theta|\sigma^2); \lambda_1/\sigma^2, \lambda_2/\sigma^2)$. Hence, we update jointly and independently the $\delta_j$ by setting $\delta_j = 1$ with probability $e^r/(1 + e^r)$, where

$$r = \log \frac{q}{1-q} + \frac{1}{2} \log(2\pi \gamma)$$

$$- \left[ (\nabla \ell_n(\theta|\sigma^2))_j d_j + \log Z + \frac{\alpha \lambda_1 |d_j| + 0.5(1-\alpha) \lambda_2 d_j^2}{\sigma^2} + \frac{d_j^2 - 2\theta_j d_j}{2\gamma} \right].$$
4.1.2. Updating $\theta$. Given $\delta$ and $\phi$, we update the components of $\theta$ using a mix of an independence Metropolis sampler, and a Metropolis Adjusted Langevin algorithm (MaLa). The MaLa strategy needs some motivation. Although its definition might perhaps suggest otherwise, the function $P_{n,\gamma}$ in \eqref{5} is actually differential (Bauschke and Combettes (2011) Proposition 12.29) and for all $\theta, H \in \mathbb{R}^d$, 

$$
\nabla_\theta P_{n,\gamma}(\theta|\sigma^2, \delta, \phi) \cdot H = \frac{1}{\gamma} \langle \theta - \operatorname{Prox}^{P_n}_\gamma(\theta|\delta, \phi), H \rangle.
$$

And since $\ell_n$ is twice continuously differentiable in this example, the expression \eqref{6} shows that $h_{n,\gamma}$ is in fact differential and for all $\theta, H \in \mathbb{R}^d$, 

$$
\nabla_\theta h_{n,\gamma}(\theta|\sigma^2, \delta, \phi) \cdot H = \langle \nabla \ell_n(\theta|\delta, \phi), H \rangle - \gamma \langle \nabla \ell_n(\theta|\delta, \phi), \nabla^{(2)} \ell_n(\theta|\delta, \phi) \cdot H \rangle + \frac{1}{\gamma} \langle \theta - \gamma \nabla \ell_n(\theta|\delta, \phi) - J_\gamma(\theta|\delta, \phi), \left( I_d - \gamma \nabla^{(2)} \ell_n(\theta|\delta, \phi) \right) \cdot H \rangle
$$

$$
= \frac{1}{\gamma} \langle \theta - J_\gamma(\theta|\delta, \phi), \left( I_d - \gamma \nabla^{(2)} \ell_n(\theta|\delta, \phi) \right) \cdot H \rangle.
$$

To avoid dealing with second order derivatives, and since $\gamma$ is typically small, we make the approximation $I_d - \gamma \nabla^{(2)} \ell_n(\theta|\delta, \phi) \approx I_d$, and therefore, we approximate $\nabla_\theta h_{n,\gamma}(\theta|\delta, \phi)$ by 

$$
G_\gamma(\theta|\delta, \phi) \overset{\text{def}}{=} \frac{1}{\gamma} (\theta - J_\gamma(\theta|\delta, \phi)), \quad \text{and} \quad \tilde{G}_\gamma(\theta|\delta, \phi) \overset{\text{def}}{=} \frac{c}{c \vee \|G_\gamma(\theta|\delta)\|} G_\gamma(\theta|\delta, \phi), \quad (17)
$$

for a positive constant $c$. The function $\tilde{G}_\gamma$ is introduced for further stability, in the spirit of the truncated Metropolis adjusted Langevin algorithm (see e.g. [Atchadé (2006)]). Hence, given $\delta$ and $\phi$, one can update the components of $\theta$ using a Metropolized-Langevin-type algorithm where the drift function is given by the corresponding components of $G_\gamma$. This algorithm is similar to the proximal MaLa of [Peyré (2013)].

However, when $\delta_j = 0$, the corresponding component of $G_\gamma(\theta|\delta, \phi)$ is $\theta_j/\gamma$ and is typically very large and not very informative (particularly for $\gamma$ small). To deal with this, we use the following strategy. We update jointly the components $\theta_j$ for which $\delta_j = 1$ using the MaLa algorithm outlined above. Then, we group together all the components for which $\delta_j = 0$ and we update them jointly using an independence Metropolis sampler. The proposal density of the Independence Metropolis sampler is built by approximating $J_\gamma(\theta|\delta)$ by $\operatorname{Prox}^{P_n}_\gamma(\theta|\delta)$. This approximation makes sense because, for $\gamma \approx 0$, $J_\gamma(\theta|\delta) = \operatorname{Prox}^{P_n}_\gamma(\theta - \gamma \nabla \ell_n(\theta|\sigma^2))|\delta \approx \operatorname{Prox}^{P_n}_\gamma(\theta|\delta)$.

To explain the detail of the independence sampler, let $\theta_\delta = (\theta_j, J : \text{s.t. } \delta_j = 1)$, $u = (\theta_j, J : \text{s.t. } \delta_j = 0)$, and let us represent $\theta$ by the pair $(\theta_\delta, u)$. Let $h_{n,\gamma}(\theta_\delta, u|\sigma^2, \phi)$ be the function obtained by replacing $J_\gamma(\theta_\delta, u|\delta)$ by $\operatorname{Prox}^{P_n}_\gamma(\theta_\delta, u|\delta)$ in \eqref{15}. Because,
Prox\(^{P_n}_{\gamma}(\theta, u|\delta)\) does not actually depend on \(u\), we have

\[
\tilde{h}_{n,\gamma}(\theta, u|\sigma^2, \phi) = \ell_n(\theta|\sigma^2) + \langle \nabla \ell_n(\theta|\sigma^2), \text{Prox}_{\gamma}^{P_n}(\theta, u|\delta) \rangle
\]
\[
+ \frac{1}{2\gamma} \| \text{Prox}_{\gamma}^{P_n}(\theta, u|\delta) - \theta \|_2^2 + \text{const.}
\]
\[
= \frac{1}{2\sigma^2} \| y - X_\delta \theta - X_{\delta^c} u \|_2^2
\]
\[
- \frac{1}{\sigma^2} \langle y - X_\delta \theta - X_{\delta^c} u, X(\text{Prox}_{\gamma}^{P_n}(\theta, u|\delta) - \delta \cdot \theta) + X_{\delta^c} u \rangle
\]
\[
+ \frac{1}{2\gamma} \| u \|_2^2 + \text{const.}
\]

It is then easy to see that \(u \mapsto e^{-\tilde{h}_{n,\gamma}(\theta, u|\sigma^2, \phi)}\) is proportional to the density of the Gaussian distribution

\[N\left(\frac{\gamma}{\sigma^2} \Sigma X_{\delta^c} X(\text{Prox}_{\gamma}^{P_n}(\theta|\delta) - \delta \cdot \theta), \gamma \Sigma\right),\]

where \(\delta^c\) is the vector \(1 - \delta\), and for any \(\delta \in \Delta\), \(X_\delta \in \mathbb{R}^{n \times \|\delta\|}\) denote the sub-matrix of \(X\) obtained by selecting the columns for which \(\delta_j = 1\). Notice that under the assumption \(\gamma \leq \frac{\sigma^2}{\lambda_{\text{max}}(X'X)}\), the matrix \(\Sigma\) is always positive definite. The acceptance probability of this independence sampler is

\[
\min \left[ 1, \frac{\exp \left( h_{n,\gamma}(\theta, u'|\sigma^2, \phi) - \tilde{h}_{n,\gamma}(\theta, u'|\sigma^2, \phi) \right)}{\exp \left( h_{n,\gamma}(\theta, u|\sigma^2, \phi) - \tilde{h}_{n,\gamma}(\theta, u|\sigma^2, \phi) \right)} \right].
\]

We found this independence sampler to be extremely efficient, with an acceptance probability typically above 90%. Notice however that this independence sampler requires a Cholesky decomposition to draw samples from the Gaussian distribution. Hence for \(d\) very large (\(d > 10,000\)), the cost of the Cholesky can become prohibitive. In such cases, one should revert to a sampling strategy based solely on the MaLa update described above.

### 4.1.3. Updating \(\phi = (q, \lambda_1, \lambda_2)\)

We update \(q \sim \text{Beta}(\|\delta\|_1 + 1, d - \|\delta\|_1 + 1)\), and we update \((\lambda_1, \lambda_2)\) jointly using a Random Walk Metropolis algorithm with Gaussian proposal. For improved mixing, we adaptively tune the scale parameter of the proposal density.

### 4.2. Simulation results and comparison with STMaLa

We illustrate the method with a simulated data example. All the computations in this example were done using \texttt{Matlab} 7.14 on a 2.8 GHz Quad-Core \texttt{Mac Pro} with 24 GB of 1066 DDR3 Ram.

We set \(n = 200, p = 500\) and we generate the design matrix \(X\) by simulating the rows of \(X\) independently from a Gaussian distribution with correlation \(\rho^{|j-i|}\) between
components $i$ and $j$. We set $\rho = 0.9$. Using $X$, we general the outcome $y = X\theta_* + \sigma \epsilon$, with $\sigma = 1$. We consider two scenarios for $\theta_*$. In the first scenario (SCENARIO 1), 10 components of $\theta_*$ are randomly selected, and we fill those components with draws from the uniform distribution $U(-v - 1, -v) \cup (v, v + 1)$, with $v = 1$. All other components are set to zero. In the second scenario (SCENARIO 2), we set $\theta_{*,1:5} = (1, 1.25, 1.5, 1.75, 2)$, and $\theta_{*,201:255} = -(1, 1.25, 1.5, 1.75, 2)$. All other components are set to zero. Notice that, because of the strong correlation between nearby regressors in the design matrix $X$, the second scenario leads to a more challenging posterior distribution.

First, we investigate the effect of the parameter $\gamma$. For these simulations, we assume $\sigma^2$ known. We set $\gamma = \gamma_0 \lambda_{\text{max}}(X'X)/\sigma^2$ as prescribed by $(16)$ with two choices of $\gamma_0$: $\gamma_0 = 0.25$, and $\gamma_0 = 0.01$.

We compare these two samplers to the STMaLa sampler of Schreck et al. (2013). The comparison is slightly tricky because STMaLa uses a different prior, namely a Gaussian “slab” prior. However, we expect both posterior distribution on $(\delta, \theta)$ to be close, and we expect $(\delta_*, \theta_*)$ to be at the center of both distributions.

For the STMaLa, we use the Matlab code provided online by the authors, with the default setting. Unlike our approach, this sampler requires the true value of the sparsity parameter $q$, which we provide. We also edit their code to return the summary statistics presented below.

We evaluate the mixing of these samplers by computing the following two metrics along the MCMC iterations: the relative error and the $F$-score (to evaluate structure recovery), defined respectively as

$$E(k) = \frac{\|\theta(k) - \theta_*\|}{\|\theta_*\|}, \quad \text{and} \quad F(k) = \frac{2 \times \text{SEN}(k) \text{PREC}(k)}{\text{SEN}(k) + \text{PREC}(k)},$$

where

$$\text{SEN}(k) = \frac{\sum_{j=1}^{d} 1\{|\theta_j(k)| > 0\} 1\{|\theta_*| > 0\}}{\sum_{j=1}^{d} 1\{|\theta_*| > 0\}} \quad \text{and} \quad \text{PREC}(k) = \frac{\sum_{j=1}^{d} 1\{|\theta_j(k)| > 0\} 1\{|\theta_*| > 0\}}{\sum_{j=1}^{d} 1\{|\theta_j(k)| > 0\}}.$$  \hspace{1cm} (18)

In order to account for the computing time, and for better comparison, we plot these metrics, not as function of the iterations $k$, but as function of the computing time needed to reach iteration $k$. For further stability in the comparison, we repeat all the samplers 30 times and average the two metrics and the computing times over these 30 replications.
All the chains are initialized by setting all components of $\theta^{(0)}$ (and $\delta^{(0)}$) to zero. For sparse regression problems, this is a very reasonable initialization.

We run the samplers for a number of iterations that depends on $\theta^\star$. In the case where $\theta^\star$ is randomly generated (SCENARIO 1), we run the newly proposed sampler for 10,000, and we run STMaLa for 80,000 iterations. In the second scenario (SCENARIO 2), we run our proposed sampler for 60,000, and we run STMaLa for 400,000 iterations.

Figure 1 and 2 present the results. First, we observe that that $\gamma_0 = 0.25$ mixes significantly better than $\gamma_0 = 0.01$. We notice also that $\hat{\Pi}_{n,\gamma}$ approximates $(\theta^\star, \delta^\star)$ only slightly better when $\gamma_0 = 0.01$ compared to $\gamma_0 = 0.25$. This suggests that in practice one should not set $\gamma_0$ too small. We found that $\gamma_0 \in (0.1, 0.25)$ behaves reasonably well. Finally, we observe from Figure 2 that STMaLa failed to converge in 400,000 iterations in SCENARIO 2.

We also look at the usual sample path mixing of the proposed sampler by plotting the trace plot, histogram, and the autocorrelation plot from a single run of the sampler (Figure 3-4). Here, we consider the scenario SCENARIO 1 with $v = 2$, and we set $\gamma_0 = 0.25$. We look at the MCMC output $\{\theta_j^{(k)}, k \geq 0\}$, for one component $j$ for which $\delta_j = 0$, and for one component $j$ for which $\delta_j = 1$ (Figure 4). We also present the same types of plots for the parameter $q$ (the sparsity parameter), and for the regularization parameters $\lambda_1, \lambda_2$ (Figure 3). From the sample path perspective, these
two figures suggest that the sampler has a good mixing. Figure 4 also shows that for the components \( j \) such that \( \delta_j = 0 \), the marginal distribution of \( \theta_j \) is centered around 0 with a small variance, as expected. We also observe with interest that in Figure 3, the values of the regularization \( \lambda_1 \) needed to produce the sparse models is typically small. This contrasts with the frequentist setting where large values of the regularization parameter are needed for sparse models, and is consistent with some of the theoretical findings by Castillo et al. (2014).

4.3. **Empirical Bayes implementation and further experimentation.** An important limitation of the methodology presented above is that \( \sigma^2 \) is assumed known, which is rarely the case in practice. Because the Lipschitz constant \( L \) in this problem depends on \( \sigma^2 \), it is not immediately clear how to implement an efficient, fully Bayesian approach that integrates \( \sigma^2 \), and still use the Moreau-Yosida approximation. As an alternative, we explore an empirical Bayes solution whereby \( \sigma^2 \) is estimated from data. Reid et al. (2013) has recently conducted an extensive simulation study to compare various methods for estimating the noise level in lasso. This paper recommends the estimator

\[
\hat{\sigma}_n^2 = \frac{1}{n - \hat{\delta}_\lambda_n} \sum_{i=1}^{n} \left( y_i - x_i \hat{\beta}_\lambda_n \right)^2,
\]
where $\hat{\beta}_\lambda$ is the lasso estimate at regularization level $\lambda$, and $\lambda_n$ is selected by 10-fold cross-validation, and where $\hat{s}_{n\lambda}$ is the number of non-zeros components of $\hat{\beta}_{n\lambda}$. In the cross-validation, we choose $\lambda_n$ as the value of $\lambda$ that minimizes the MSE. This leads to the empirical Bayes Moreau-Yosida posterior approximation $\Pi_{n,\gamma}(\cdot|y, \hat{\sigma}_n^2)$.

We do a simulation using a semi-real dataset to compare the distributions $\Pi_{n,\gamma}(\cdot|y, \hat{\sigma}_n^2)$ and $\Pi_{n,\gamma}(\cdot|y, \sigma^2)$ (where $\sigma^2 = 1$ is the true value of $\sigma^2$). We use the colon dataset (Buhlmann and Mandozzi (2014)) downloaded from http://stat.ethz.ch/~dettling/bagboost.html. The data gives microarray gene expression levels for 2,000 genes for $n = 62$ patients in a colon cancer study. We randomly select a subset of $p = 1,000$ variables to form a design matrix $X \in \mathbb{R}^{62 \times 1,000}$. Following Buhlmann and Mandozzi (2014), we normalize each column of $X$ to have mean zero and variance unity. We simulate a sparse signal vector $\theta^\star \in \mathbb{R}^p$ with $s = 5$ non-zeros components, and where the non-zeros components are drawn from $U(-\nu - 1, -\nu) \cup (\nu, \nu + 1)$. We consider two scenarios: $\nu = 1$ and $\nu = 3$. Using $X$ and $\theta^\star$, we generate $y = X\theta^\star + \sigma \epsilon$, with $\sigma = 1$, and $\epsilon \sim \mathcal{N}(0, I_n)$.

We use our proposed algorithm in Section 4.1 to sample from $\Pi_{n,\gamma}(\cdot|y, \hat{\sigma}_n^2)$ and $\Pi_{n,\gamma}(\cdot|y, \sigma^2)$, with $\gamma_0 = 0.25$. We evaluate the samplers along the same metrics $E$ and
Figure 4. Trace plot, histogram, and autocorrelation plot, from one MCMC run, using $\gamma_0 = 0.25$. Top row is for a component $j$ for which the true value of $\delta_j$ is 0. Bottom row, true value of $\delta_j$ is 1.

\[ F \]. We average the results over 30 replications\(^3\) of the samplers, where each sampler is run for 50,000 iterations. The result is on Figures 5-6. We notice that when $v = 1$, none of the two posterior distributions contains much information on $\theta_*$. When the signal is strong ($v = 3$), the empirical Bayes posterior distribution performs well, but as expected, underperforms the posterior distribution with known variance. It is well known that inference based on empirical Bayes posterior distributions can be inexact due to the additional uncertainty in estimating $\sigma^2$ (Laird and Louis (1987)). Correcting this bias in the present high-dimensional framework is an important problem that is beyond the scope of this work.

5. Application to Gaussian graphical model selection

In this section we use a Gaussian graphical model to illustrate the usefulness of the quasi-Bayesian framework and the Moreau-Yosida approximation beyond the linear model setting. To keep the length of the paper reasonable, we focus mainly on the

\(^3\)here only $X$ and $\theta_*$ are kept fixed. For each replication, the data set $y$ is re-simulated, and $\sigma^2_n$ is re-estimated.
Figure 5. Comparing the empirical Bayes and exact posterior distributions, when $v = 1$. Based on 30 MCMC replications each run for $5 \times 10^5$ iterations. Average estimate of $\sigma^2$ over the 30 replications is 1.20. The curves are sub-sampled to improve readability.

Figure 6. Comparing the empirical Bayes and exact posterior distributions, when $v = 3$. Based on 30 MCMC replications each run for $5 \times 10^5$ iterations. Average estimate of $\sigma^2$ over the 30 replications is 0.74. The curves are sub-sampled to improve readability.
applicability of the methods. Further investigations, including an extension to non-Gaussian graphical models will be reported elsewhere.

We denote \( \mathcal{M}_p \) the space of real-valued \( p \times p \) symmetric matrices, and \( \mathcal{M}_p^+ \) its subset of positive definite matrices. For \( \mathcal{K} \in \mathcal{M}_p^+ \), let \( f_{\mathcal{K}} \) be the density (on \( \mathbb{R}^p \)) of the Gaussian distribution with precision matrix \( \mathcal{K} \). The precision matrix \( \mathcal{K} \) encodes the conditional independence structure among the \( p \) variables, and is the parameter of interest. In particular for \( i \neq j \), \( \mathcal{K}_{ij} = 0 \) means that if \( (Y_1, \ldots, Y_p) \sim f_{\mathcal{K}} \), then \( Y_i \) and \( Y_j \) are conditionally independent given all other variables. The random variables \( (Y_1, \ldots, Y_p) \) can be represented by an undirected graph where there is an edge between \( i \) and \( j \) when \( \mathcal{K}_{ij} \neq 0 \). This type of models are very useful in practice to tease out direct and indirect dependence structures between sets of random variables. The estimation of high-dimensional precision matrices has generated an extensive literature in recent years (Drton and Perlman (2004); Meinshausen and Buhlmann (2006); Yuan and Lin (2007); d’Aspremont et al. (2008); Friedman et al. (2008); Ravikumar et al. (2011) and the references therein).

We consider the Bayesian approach to this problem. Suppose that \( y^{(1)}, \ldots, y^{(n)} \in \mathbb{R}^p \) are realizations of i.i.d. random variables with conditional distribution \( f_{\mathcal{K}} \), given a parameter \( \mathcal{K} \in \mathcal{M}_p^+ \). Given a prior distribution \( \Pi \) for \( \mathcal{K} \) on \( \mathcal{M}_p^+ \), the posterior distribution of \( \mathcal{K} \) is

\[
\Pi_n(A|y^{(1)}, \ldots, y^{(n)}) = \frac{\int_A \prod_{i=1}^n f_{\mathcal{K}}(y^{(i)}) \Pi(d\mathcal{K})}{\int_{\mathcal{M}_p^+} \prod_{i=1}^n f_{\mathcal{K}}(y^{(i)}) \Pi(d\mathcal{K})}, \quad A \subseteq \mathcal{M}_p^+.
\]

Most of the existing literature on Bayesian precision matrix estimation focuses on weak sparsity (Wang (2012); Khondker et al. (2013)). For exact sparsity, a prior distribution on the joint space of structures and parameters is needed. More precisely, we need a prior distribution \( \Pi \) on \( \mathcal{S}_p \times \mathcal{M}_p^+ \) of the form

\[
\Pi(\delta, A) = \pi_{\delta} \Pi(A|\delta), \quad \delta \in \mathcal{S}_p, \quad A \subseteq \mathcal{M}_p^+,
\]

where \( \mathcal{S}_p \) is the set of precision matrix structures (that is the subset of \( \mathcal{M}_p \) with zero-one entries, and diagonal identically one), \( \{\pi_{\delta}, \delta \in \mathcal{S}_p\} \) is a probability distribution on \( \mathcal{S}_p \), and for a probability measure \( \Pi(\cdot|\delta) \) on \( \mathcal{M}_p^+ \) such that \( \Pi(\{\mathcal{K} \in \mathcal{M}_p^+: \mathcal{K}_{ij} = 0, \text{ whenever } \delta_{ij} = 0\}|\delta) = 1 \). However, the requirement that the prior \( \Pi(\cdot|\delta) \) concentrates on the set \( \{\mathcal{K} \in \mathcal{M}_p^+: \mathcal{K}_{ij} = 0, \text{ whenever } \delta_{ij} = 0\} \) typically yields prior distributions with intractable normalizing constants, and for which practical implementation through MCMC is complicated. Banerjee and Ghosal (2013) recently tackles this problem, building \( \Pi(\cdot|\delta) \) from a product of Laplace and exponential priors. However this paper seems to have ignored the intractable normalizing constants issue. Another well-established route to build these priors is via the G-Wishart priors.
(Wang and Li (2012); Peterson et al. (2015) and the references therein), which leads to the same intractable normalizing constants issues. Wang and Li (2012) develops an MCMC sampler to deal with G-Wishart priors, based on the double-Metropolis of Liang (2010). However, as pointed out in Lyne et al. (2013), the double Metropolis algorithm does not possess the correct invariant distribution, and a careful analysis of its approximation errors is yet to be done.

A quasi-Bayesian approach can be used to circumvent some of the difficulties discussed above. For $K \in \mathcal{M}_p^+$, and $j \in \{1, \ldots, p\}$, if $(Y_1, \ldots, Y_p) \sim f_K$, the conditional distribution of $Y_j$ given the remaining random variables is

$$N(\mu_j, \sigma_j^2), \text{ where } \mu_j = -\sigma_j^2 \sum_{k \neq j} K_{jk} Y_k, \quad \sigma_j^2 = \frac{1}{K_{jj}}.$$  

Notice that these conditional distributions do not require the positive definiteness of $K$, only the positiveness of its diagonal elements. Hence for parameter $\vartheta \in \mathcal{M}_p$, with zero-diagonal, and $\sigma^2 = (\sigma_1^2, \ldots, \sigma_p^2) \in \mathbb{R}_+^p$, where $\mathbb{R}_+ \overset{\text{def}}{=} (0, \infty)$, we consider the quasi-density

$$q_{\sigma^2, \vartheta}(y^{(1)}, \ldots, y^{(n)}) \overset{\text{def}}{=} \prod_{i=1}^n \prod_{j=1}^p \sqrt{\frac{1}{2\pi \sigma_j^2}} \exp \left(-\frac{1}{2\sigma_j^2} \left( y_j^{(i)} + \sigma_j^2 \sum_{k \neq j} \vartheta_{jk} y_k^{(i)} \right)^2 \right). \quad (19)$$

The parameter $\vartheta$ corresponds to $K$ with its diagonal set to zero, and viewed as element of $\mathcal{M}_p$. It is well known that maximizing (19) gives a reasonable estimator of $\sigma^2$ and $\vartheta$. This is the pseudo-likelihood function of Besag (1974). It is also possible to parametrize (19) in terms of the partial correlations, which is the approach employed by Peng et al. (2009). However, a parametrization in terms of partial correlations is not very convenient in the Bayesian framework, since partial correlations are constrained to the interval $[-1, 1]$, and therefore require prior distributions that reflect this constraint.

We consider a quasi-Bayesian approach whereby the likelihood function is replaced by (19). We will assume that $\sigma^2$ is known, and we focus on the estimation of the structure of $\vartheta$. In order to use the notation of Section 2 we further reparametrize the problem. We set $\Theta = \mathbb{R}^d$, where $d = p(p-1)/2$. Let $\mathcal{M}_{p,0}$ denote the elements of $\mathcal{M}_p$ with zero diagonal. For $\theta \in \mathcal{M}_{p,0}$, we denote $\text{Vec}(\theta)$ the element of $\mathbb{R}^d$ obtained by concatenating column by column, left to right, the below-and-off-diagonal elements of $\theta$. We also denote $L : \mathbb{R}^d \rightarrow \mathcal{M}_{p,0}$ the inverse map of $\text{Vec}$: $\text{Vec}(L(u)) = u$, for $u \in \mathbb{R}^d$. Hence $L(u)$ is symmetric, with zero diagonal, and $(L(u))_{2p,1} = u_{1:(p-1)}$, $(L(u))_{3p,2} = (u_{p:(2p-3)}$, etc... For $1 \leq j \leq p$, let $y_j = (y_j^{(1)}, \ldots, y_j^{(n)})' \in \mathbb{R}^n$ be the data collected on the $j$-th node. Let $y \overset{\text{def}}{=} [y_1, \ldots, y_p] \in \mathbb{R}^{n \times p}$. Now, for a vector
\[ \theta \in \Theta, \text{ and } \sigma^2 \in \mathbb{R}^p_+, \] can be re-written as
\[
q_{\sigma^2, \theta}(y) = \prod_{j=1}^{p} \left( \frac{1}{2\pi \sigma_j^2} \right)^{n/2} \exp \left( -\frac{1}{2} \sum_{j=1}^{p} \sigma_j^2 \left\| y \left( L(\theta) + \text{diag} \left( \frac{1}{\sigma^2} \right) \right) \right\|^2 \right),
\]
where \( A_j \) denotes the \( j \)-th column of the matrix \( A \), and \( \text{diag}(1/\sigma^2) \) is the the \( \mathbb{R}^{p \times p} \) diagonal matrix with diagonal elements \( (1/\sigma_1^2, \ldots, 1/\sigma_p^2) \).

We build a similar prior distribution as in the linear regression problem. Given \( \delta \in \Delta \), \( q \in (0, 1) \), and given hyper-parameters \( \lambda_1, \lambda_2 \), we assume that the components of \( \theta \) are independent, and for \( 1 \leq k \leq d \),
\[
\theta_k|\delta, \alpha, \lambda_1, \lambda_2 \sim \begin{cases} 
\text{Dirac}(0) & \text{if } \delta_k = 0 \\
p(\cdot | \lambda_1 w_k, \lambda_2 w_k) & \text{if } \delta_k = 1 
\end{cases},
\]
where the density \( p \) is as in (11). The weight \( w_k \) is used to modulate the penalization across components. Following Sun and Zhang (2013), we choose \( w_k \) as follows. For \( \theta \in \mathbb{R}^d \), let \((i_k, j_k)\) denote the coordinate of \( \theta_k \) in the matrix \( L(\theta) \). Although other choices are possible, we set
\[
w_k = \sqrt{\text{Var} \left( y_{i_k}^{(1:n)} \right) \text{Var} \left( y_{j_k}^{(1:n)} \right)}
\]
\[
= \frac{1}{n} \left[ \sum_{t=1}^{n} \left( y_{i_k}^{(t)} - \frac{1}{n} \sum_{q=1}^{n} y_{i_k}^{(q)} \right)^2 \right] \sqrt{\sum_{t=1}^{n} \left( y_{j_k}^{(t)} - \frac{1}{n} \sum_{q=1}^{n} y_{j_k}^{(q)} \right)^2}.
\]

As above, we assume that the sparsity parameter \( q \) is a uniform random variable \( q \sim \mathcal{U}(0, 1) \), and given \( q \), the components of \( \delta \) are independent with a Bernoulli distribution \( \text{Ber}(q) \). Finally, we assume that \( \lambda_1 \) and \( \lambda_2 \) are uniformly distributed on \( \mathcal{U}(a, M) \), for \( 0 < a < M < \infty \). Hence for this example, \( X^{(n)} = \mathbb{R}^{n \times p} \), \( \Theta = \mathbb{R}^d \), \( \phi = (\pi, \lambda_1, \lambda_2) \), \( \Xi = (0, 1) \times (a, M) \times (a, M) \), and \( G \) is the uniform distribution on \( \Xi \). The function \( \ell_n \) and \( P_n \) in (3) are given by
\[
\ell_n(\theta|\sigma^2) = \frac{1}{2} \sum_{j=1}^{p} \sigma_j^2 \left\| y \left( L(\theta) + \text{diag} \left( \frac{1}{\sigma^2} \right) \right) \right\|^2 \quad \text{and}
\]
\[
P_n(\theta|\sigma^2, \delta, \phi) = \sum_{j=1}^{d} \delta_j \left( \log Z_j + \alpha \lambda_1 w_j | \theta_j | + \frac{(1 - \alpha) \lambda_2}{2} w_j \theta_j^2 \right) + \ell_{\Theta_k}(\theta),
\]
where \( Z_j \overset{\text{def}}{=} \tilde{Z}_\alpha(w_j \lambda_1, w_j \lambda_2) \). With \( h_n(\theta|\sigma^2, \delta, \phi) = \ell_n(\theta|\sigma^2) + P_n(\theta|\sigma^2, \delta, \phi) \), the resulting quasi-posterior distribution is
\[
\tilde{\Pi}_n(\delta, d\phi, d\theta|y, \sigma^2) \propto q^{d-|\delta|} \left( 1 - q \right)^{d-|\theta|} e^{-h_n(\theta|\sigma^2, \delta, \phi)} \mu_p(d\theta) d\phi.
\]
(20)
Since \( q_{\alpha^{2}, \sigma}(y) \leq \prod_{j=1}^{p} (1/2\pi\sigma_{j}^{2})^{n/2} \), and \( \sigma^{2} \) is assumed fixed, the following proposition is easy to prove and shows that \( \hat{\Pi}_{n} \) is a well-defined probability measure.

**Proposition 6.** For any \( \sigma^{2} \in \mathbb{R}_{+}^{p} \), \( \hat{\Pi}_{n}(\cdot|y, \sigma^{2}) \) is a well-defined probability measure on \( \Delta \times \Xi \times \Theta \).

As with the linear regression case, the Moreau-Yosida approximation of \( \hat{\Pi}_{n} \) has a density \( \tilde{\pi}_{n,\gamma} \) given by

\[
\tilde{\pi}_{n,\gamma}(\delta, \phi, \theta|y, \sigma^{2}) \propto q^{\|\delta\|_{1}}(1 - q)^{d - \|\delta\|_{1}} (2\pi\gamma)^{\|\delta\|_{1}/2} \times e^{-h_{n,\gamma}(\theta|\delta, \sigma^{2}, \phi)},
\]

where \( h_{n,\gamma} \) is given by

\[
h_{n,\gamma}(\theta|\delta, \sigma^{2}, \phi) = \ell_{n}(\theta|\sigma^{2}) + \langle \nabla \ell_{n}(\theta|\sigma^{2}), J_{\gamma} - \theta \rangle + \sum_{j=1}^{d} \delta_{j} \log Z_{j} + \alpha \lambda_{1}\|w \cdot J_{\gamma}\|_{1} + \frac{(1 - \alpha)}{2} \lambda_{2}\|w \cdot J_{\gamma}\|^{2} + \frac{1}{2}\gamma\|J_{\gamma} - \theta\|^{2},
\]

where the \( j \)-th component of \( J_{\gamma} \) is \( \delta_{j}s_{\gamma} \left( \langle \theta - \gamma \nabla \ell_{n}(\theta|\sigma^{2}) \rangle_{j} ; \lambda_{1}w_{j}, \lambda_{2}w_{j} \right) \).

**Theorem 7.** Fix \( \sigma^{2} \in \mathbb{R}_{+}^{p} \), and the dataset \( y \in \mathbb{R}^{n \times p} \). Set \( \gamma_{0} = \frac{1}{8\alpha^{2}\lambda_{\max}} \), and suppose that \( M < 2\alpha^{2}\lambda_{\max}|y'y|/(1 - \alpha) \), where \( \sigma^{2}_{j} = \max_{1 \leq j \leq p} \sigma_{j}^{2} \). Then for all \( \gamma \in (0, \gamma_{0}] \), \( \tilde{\Pi}_{n,\gamma} \) is well-defined on \( \Delta \times \Xi \times \Theta \). Furthermore, if \( d \geq 2 \), there exists a finite constant \( C_{d} \) such that for all \( \gamma \in (0, \gamma_{0}] \)

\[
\beta \left( \hat{\Pi}_{n,\gamma}, \Pi_{n} \right) \leq \sqrt{\gamma d} + \gamma \left( C_{d} + 5Ld \right).
\]

*Proof.* See Section 7.3 \( \square \)

### 5.1. Markov Chain Monte Carlo and Simulation results.

Barring the function \( \ell_{n} \), its gradient, and the weight \( w \), the function \( h_{n,\gamma} \) above is the same as in the linear regression case in (15). Hence, we proceed similarly to sample from \( \tilde{\pi}_{n,\gamma}(\delta, \theta, \phi|y, \sigma^{2}) \). The update of \( \delta \) and \( \phi \) are identical, again, barring the specific expression of \( \ell_{n} \) and the weight \( w \). The strategy for updating \( \theta \) is slightly different. Here we use only the MaLa update. Given \( \theta^{(k)} \) that we identify as \( L(\theta^{(k)}) \), we update successively each of the first \( p - 1 \) columns of \( L(\theta^{(k)}) \) using the MaLa algorithm outlined in Section 4.1.

Notice that, since \( L(\theta^{(k)}) \) is symmetric, at the end of each sweep, all the components of (the vector) \( \theta^{(k)} \) are updated twice.

We illustrate the procedure with a simulated example with \( p = 100 \), and \( n \in \{100, 500, 10^{3}, 3 \times 10^{3}\} \). We generate the data as follows. We generate a sparse matrix \( B \) (using the Matlab function `sprandn`) with an average proportion of non-zeros entries set to \( 5/p \). Then we symmetrize \( B \) by forming \( A = (B + B')/2 \). Hence, \( A \) has on average \( 10p \) non-zero entries. We strengthen the signal by adding \( 3 \times \text{sign}(A_{ij}) \)
to all non-zero entries $A_{ij}$ of $A$. Finally, to guarantee positive definiteness, we set $\theta_* = A + (\epsilon - \lambda_{\min}(A))I_p$, with $\epsilon = 1$. Using $\theta_*$, we generate the data $y^{(1)}, \ldots, y^{(n)} \sim_{i.i.d.} N(0, \theta_*^{-1})$.

Following Theorem 7, we set $\gamma = \frac{\gamma_0}{\sigma^2 \lambda_{\max}(y'y)}$, and we compare two values of $\gamma_0$: $\gamma_0 = 1$, and $\gamma_0 = 0.1$. We initialize the samplers by setting all components to zero, and we run the MCMC for $N_{\text{iter}} = 10^4$. After a burn-in of $N_{\text{iter}}/2$, we compute the average value of the precision $\text{PRE}$, and the sensitivity $\text{SEN}$ along the MCMC iterations. For further stability, we average these estimates over 30 replications of MCMC samplers. The results are presented in Table 1. For $n = 3,000$, we expect the posterior distribution to concentrate well around the true value $\theta_*$. Hence, in this case, the trace plot of the relative error $E$ and the structure recovery statistic $F$ can inform about the mixing of the MCMC, and we give this on Figure 7.

This simulation study shows that a quasi-Bayesian approach can be used to learn Gaussian graphical model in a high-dimensional setting. And the resulting quasi-posterior distribution can be well approximated by MCMC samples from its Moreau-Yosida quasi-posterior approximation. Table 1 suggests that this approach has a good sensitivity, but its false-positive rate is high, even with $n = 3,000$. Although we did not explore this here, an empirical Bayes version can be easily obtained if a good estimate of $\sigma^2$ is available (using for instance the sparse matrix inversion of Sun and Zhang (2013)).

<table>
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<th>$n$</th>
<th>$\gamma_0 = 1$</th>
<th>$\gamma_0 = 0.1$</th>
<th>$\gamma_0 = 1$</th>
<th>$\gamma_0 = 0.1$</th>
<th>$\gamma_0 = 1$</th>
<th>$\gamma_0 = 0.1$</th>
</tr>
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<td>99.8</td>
<td>99.8</td>
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<tr>
<td>500</td>
<td>11.6</td>
<td>12.5</td>
<td>23.8</td>
<td>24.1</td>
<td>35.1</td>
<td>34.4</td>
</tr>
<tr>
<td>1,000</td>
<td>53.7</td>
<td>53.8</td>
<td></td>
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</table>

**Table 1.** Table showing the sensitivity and precision (as defined in (18) for varying sample size $n$. Based on 30 MCMC replications, each MCMC run is $10^4$ iterations.

6. Further Discussion

This work uses the Moreau-Yosida regularization to derive a smooth approximation to posterior (and quasi-posterior) distributions with exact-sparsity inducing prior distributions. The methodology is very general, and can be readily used to implement Bayesian ideas with a large class of high-dimensional models (linear and generalized linear regression models, graphical models, sparse PCA, and others). However, several theoretical issues remain. We have mentioned some of these issues throughout...
Figure 7. Relative error, and structure recovery as function of computing time for the Gaussian graphical model example, with $p = 100$, $n = 3,000$. Based on 30 replications of MCMC runs of $10^4$ iterations. The curves are sub-sampled to improve readability.

the paper. One important problem that we did not directly address concerns the mixing properties of the proposed MCMC algorithms. Clearly the simulation examples performed in this work suggest that Moreau-Yosida quasi-posterior approximations possess some features that greatly facilitate simulation by MCMC. What are these features? It is the smoothness, the log-concavity? or perhaps some other features? Rigorous formulation and answers to this question will greatly advance our understanding of the method.

7. Proofs

7.1. Proof of Theorem 3. We introduce two more distances between probability measures on $\tilde{\Theta} = \Delta \times \Xi \times \Theta$. For two probability measures $\nu_1, \nu_2$ on $\Theta$, their total variation distance is defined as

$$\|\nu_1 - \nu_2\|_{tv} \overset{\text{def}}{=} \sup_{\|f\|_{\infty} \leq 1} \left| \int_{\Theta} f(\theta) \nu_1(d\theta) - \int_{\Theta} f(\theta) \nu_2(d\theta) \right|,$$

where the supremum is taken over all bounded measurable function $f : \tilde{\Theta} \to \mathbb{R}$ such that $\|f\|_{\infty} \overset{\text{def}}{=} \sup_{\theta \in \tilde{\Theta}} |f(\theta)| \leq 1$, and the Wasserstein distance between $\nu_1, \nu_2$ is defined as

$$\mathcal{W}(\nu_1, \nu_2) \overset{\text{def}}{=} \sup_{\|f\|_{\infty} \leq 1} \left| \int_{\Theta} f(\theta) \nu_1(d\theta) - \int_{\Theta} f(\theta) \nu_2(d\theta) \right|,$$
where the supremum is taken over all bounded measurable function \( f : \bar{\Theta} \rightarrow \mathbb{R} \) such that \( \| f \|_L \leq 1 \), where

\[
\| f \|_L = \sup \left\{ \frac{|f(\theta_1) - f(\theta_2)|}{d(\theta_1, \theta_2)} : \theta_1, \theta_2 \in \bar{\Theta}, \theta_1 \neq \theta_2 \right\}.
\]

It is well known that

\[
\mathcal{W}(\nu_1, \nu_2) = \inf_{\nu} \mathbb{E}_\nu \left( \bar{d}(X, Y) \right),
\]

where \((X, Y) \sim \nu\), and the infimum is taken over all probability measures \( \nu \) on the product space \( \bar{\Theta} \times \bar{\Theta} \) such that \( \nu(A \times \bar{\Theta}) = \nu_1(A) \), and \( \nu(\bar{\Theta} \times A) = \nu_2(A) \) for all measurable set \( A \subset \Theta \). We refer to Dudley (2002) Chapter 11 for further details on these metrics.

For \( \gamma > 0 \), we introduce a new probability measure \( \tilde{\Pi}_{n, \gamma} \) on \( \bar{\Theta} \) defined as

\[
\tilde{\Pi}_{n, \gamma}(\delta, d\phi, d\theta|x_n) \propto \pi_\delta G(d\phi) \left( \frac{1}{2\pi \gamma} \right)^{d-\|\delta\|_1} \exp \left( -\frac{1}{2\gamma} \sum_{j, \delta_j=0} \theta_j^2 \right) e^{-h_n(\theta|\delta, \phi)} \mu_d(d\theta).
\]

This is a well-defined probability measure under \( H_2 \) because its normalizing constant, after integrating out the Gaussian densities, is

\[
C_n \defeq \sum_\delta \pi_\delta \int_{\Xi} G(d\phi) \int_{\Theta} e^{-h_n(\theta|\delta, \phi)} \mu_{d, \delta}(d\theta)
= \sum_\delta \pi_\delta \int_{\Xi} G(d\phi) \int_{\Theta} e^{-h_n(\theta, \delta)\mu_d(d\theta)},
\]

and this is finite under \( H_2 \). We prove the theorem in the following steps. First, in Lemma 8 we bound the Wasserstein distance between the newly defined distribution \( \tilde{\Pi}_{n, \gamma} \) and \( \tilde{\Pi}_n \) as \( \gamma \rightarrow 0 \). Then in Lemma 11 we bound the total variation distance between \( \tilde{\Pi}_{n, \gamma} \) and \( \tilde{\Pi}_{n, \gamma} \), as \( \gamma \rightarrow 0 \). It is clear from their definitions that both the Wasserstein metric and the total variation metric are upper bounds for the metric \( \beta \). Hence we deduce a bound on \( \beta(\tilde{\Pi}_{n, \gamma}, \tilde{\Pi}_n) \). That \( \tilde{\Pi}_{n, \gamma} \) is well-defined is shown in Lemma 10. The proof of Lemma 10 and 11 rely on Lemma 9 which gives a comparison result between the functions \( h_n \) and \( h_{n, \gamma} \).

**Lemma 8.** Assume \( H_2 \). Then for all \( \gamma > 0 \), \( \tilde{\Pi}_n \) and \( \tilde{\Pi}_{n, \gamma} \) are well-defined probability measure on \( \Delta \times \Xi \times \Theta \), and \( \mathcal{W}(\tilde{\Pi}_{n, \gamma}, \tilde{\Pi}_n) \leq \sqrt{\gamma d} \).

**Proof.** It is clear that under \( H_2 \) the quasi-posterior distribution \( \tilde{\Pi}_n \) as defined in (2) is well-defined, and so is \( \tilde{\Pi}_{n, \gamma} \) as discussed above. We set

\[
C_n(\delta, \phi) \defeq \int_{\Theta} e^{-h_n(\theta|\delta, \phi)} \mu_{d, \delta}(d\theta), \delta \in \Delta, \phi \in \Xi.
\]
The marginal distribution of \((\delta, \phi)\) in the quasi-posterior \(\Pi_n\) is given by
\[
\Pi_n(\delta, \phi|x_n) \propto C_n(\delta, \phi)\pi_\delta G(d\phi),
\]
and the conditional distribution of \(\theta\) given \(\delta, \phi\) is
\[
\Pi_n(d\theta|x_n, \delta, \phi) = \frac{e^{-h_n(\theta|\delta, \phi)}\mu_{d,\delta}(d\theta)}{C_n(\delta, \phi)}.
\]

We build two sets of random variables on \(\Theta\) as follows. First we generate \((\eta, V) \in \Delta \times \Xi\) from the marginal distribution \(\Pi_n(\delta, d\phi|x_n)\), and we generate independently \(Z_{1:d} \sim i.i.d. \ N(0, \gamma)\). Given \((\eta, V, Z)\), we generate \(\vartheta\) and \(\vartheta^{(\gamma)}\) in \(\Theta\) as follows. First, we generate \(\vartheta \sim \Pi_n(d\theta|x_n, \eta, V)\), and we generate \(\vartheta^{(\gamma)} \in \Theta\) as follows: for each \(j \in \{1, \ldots, d\}\), we set \(\vartheta^{(\gamma)}_j = \vartheta_j\) if \(\eta_j = 1\), and \(\vartheta^{(\gamma)}_j = Z_j\), if \(\eta_j = 0\). By construction, it is clear that \((\eta, V, \vartheta) \sim \Pi_n\). It is also not hard to see that \((\eta, V, \vartheta^{(\gamma)}) \sim \Pi_{n,\gamma}\) as defined in (24). Hence, using (23),
\[
W(\Pi_{n,\gamma}, \Pi_n) \leq \mathbb{E} \left[ d \left( (\eta, V, \vartheta), (\eta, V, \vartheta^{(\gamma)}) \right) \right] = \mathbb{E} \left[ \|\vartheta - \vartheta^{(\gamma)}\| \right].
\]
By conditioning on \(\eta\), and using Jensen’s inequality, we deduce that
\[
W(\Pi_{n,\gamma}, \Pi_n) \leq \sum_{\delta \in \Delta} \mathbb{P}(\eta = \delta)\mathbb{E}^{1/2} \left[ \|\vartheta^{(\gamma)} - \vartheta\|^2 | \eta = \delta \right] = \sum_{\delta \in \Delta} \mathbb{P}(\eta = \delta) \left\{ \sum_{j: \delta_j = 0} \mathbb{E}(Z_j^2 | \eta = \delta) \right\}^{1/2} = \sqrt{\gamma} \sum_{\delta \in \Delta} \mathbb{P}(\eta = \delta) (d - \|\delta\|_1)^{1/2} \leq \sqrt{\gamma}d.
\]

Lemma 9. Assume \(\Pi_{n,\gamma}\) and fix \((\delta, \phi) \in \Delta \times \Xi\). For all \(\theta \in \Theta\),
\[
h_n(\theta \cdot \delta|\phi) + \frac{1}{2\gamma} \|\theta - \theta \cdot \delta\|^2 \geq h_{n,\gamma}(\theta|\delta, \phi) \geq h_n(\theta \cdot \delta|\phi)
\]
\[
+ \left( \frac{1}{2\gamma} - L - \gamma L^2 \right) \|\theta - \theta \cdot \delta\|^2 - \gamma \| \nabla \ell_n(\theta \cdot \delta|\phi) + g_n(\theta \cdot \delta|\phi) \|^2. \tag{27}
\]
It follows in particular that for all \(\theta \in \Theta\), \(h_{n,\gamma}(\theta|\delta, \phi) \uparrow h_n(\theta|\delta, \phi)\), as \(\gamma \downarrow 0\).

Proof. From the definition we have
\[
h_{n,\gamma}(\theta|\delta, \phi) = \min_{u \in \Theta} \left[ \ell_n(\theta|\phi) + \langle \nabla \ell_n(\theta|\phi), u - \theta \rangle + P_n(u|\delta, \phi) + \frac{1}{2\gamma} \|u - \theta\|^2 \right]
\]
\[
\leq \ell_n(\theta|\phi) + P_n(\theta \cdot \delta|\phi) + \langle \nabla \ell_n(\theta|\phi), \theta \cdot \delta - \theta \rangle + \frac{1}{2\gamma} \|\theta - \theta \cdot \delta\|^2.
\]
By convexity of $\ell_n$, $\ell_n(\theta|\phi) + \langle \nabla \ell_n(\theta|\phi), \theta - \delta - \theta \rangle \leq \ell_n(\theta - \delta|\phi)$, which proves the first inequality in (27). To prove the second inequality, we start by using again the convexity of $\ell_n$ to write

$$\ell_n(\theta|\phi) \geq \ell_n(\theta - \delta|\phi) + \langle \nabla \ell_n(\theta - \delta|\phi), \theta - \theta - \delta \rangle.$$ 

Hence

$$\ell_n(\theta|\phi) + \langle \nabla \ell_n(\theta|\phi), J_\gamma(\theta|\delta, \phi - \theta) \rangle \geq \ell_n(\theta - \delta|\phi) + \langle \nabla \ell_n(\theta - \delta|\phi) - \nabla \ell_n(\theta|\phi), \theta - \theta - \delta \rangle + \langle \nabla \ell_n(\theta|\phi), J_\gamma(\theta|\delta, \phi - \theta - \delta) \rangle.$$

Using the Lipschitz assumption on $\nabla \ell_n$, it follows that

$$\ell_n(\theta|\phi) + \langle \nabla \ell_n(\theta|\phi), J_\gamma(\theta|\delta, \phi - \theta) \rangle \geq \ell_n(\theta - \delta|\phi) - L\|\theta - \theta - \delta\|^2 + \langle \nabla \ell_n(\theta|\phi), J_\gamma(\theta|\delta, \phi - \theta - \delta) \rangle.$$ (28)

By $H1$ and $H2$, $P_n(\cdot|\delta, \phi)$ is convex, and $g_n(\theta - \delta|\delta, \phi)$ belongs to the sub-differential of $P_n(\cdot|\delta, \phi)$ at $\theta - \delta$, and we have

$$P_n(J_\gamma(\theta|\delta, \phi)|\delta, \phi) \geq P_n(\theta - \delta|\delta, \phi) + \langle g_n(\theta - \delta|\delta, \phi), J_\gamma(\theta|\delta, \phi - \theta - \delta) \rangle.$$ (29)

(28)-(29) together with the expression (7) of $h_{n,\gamma}$ imply that

$$h_{n,\gamma}(\theta|\delta, \phi) \geq h_n(\theta - \delta|\delta, \phi) - L\|\theta - \theta - \delta\|^2$$

$$+ \langle \nabla \ell_n(\theta|\phi) + g_n(\theta - \delta|\delta, \phi), J_\gamma(\theta|\delta, \phi - \theta - \delta) \rangle + \frac{1}{2\gamma}\|\theta - J_\gamma(\theta|\delta, \phi)\|^2.$$

Since $J_\gamma(\theta|\delta, \phi) \in \Theta_\delta$, we can split $\|\theta - J_\gamma(\theta|\delta, \phi)\|^2$ as $\|\theta - \theta - \delta\|^2 + \|\theta - \delta - J_\gamma(\theta|\delta, \phi)\|^2$. We use this in the last inequality to conclude that

$$h_{n,\gamma}(\theta|\delta, \phi) \geq h_n(\theta - \delta|\delta, \phi) + \left(\frac{1}{2\gamma} - L\right)\|\theta - \theta - \delta\|^2$$

$$+ \langle \nabla \ell_n(\theta|\phi) + g_n(\theta - \delta|\delta, \phi), J_\gamma(\theta|\delta, \phi - \theta - \delta) \rangle + \frac{1}{2\gamma}\|J_\gamma(\theta|\delta, \phi - \theta - \delta)\|^2$$

$$\geq h_n(\theta - \delta|\delta, \phi) + \left(\frac{1}{2\gamma} - L\right)\|\theta - \theta - \delta\|^2 - \gamma\|\nabla \ell_n(\theta|\phi) + g_n(\theta - \delta|\delta, \phi)\|^2.$$ 

We further split $\nabla \ell_n(\theta|\phi)$ as $\nabla \ell_n(\theta|\phi) - \nabla \ell_n(\theta - \delta|\phi) + \nabla \ell_n(\theta - \delta|\phi)$ and use the Lipschitz property of $\nabla \ell_n$ to get the bound

$$h_{n,\gamma}(\theta|\delta) \geq \left(\frac{1}{2\gamma} - L - \gamma L^2\right)\|\theta - \theta - \delta\|^2 - \gamma\|\nabla \ell_n(\theta - \delta|\phi) + g_n(\theta - \delta|\delta, \phi)\|^2,$$

as claimed.

It is obvious from its definition that $h_{n,\gamma}(\theta|\delta, \phi)$ is non-decreasing as $\gamma \downarrow 0$. If $\theta \notin \Theta_\delta$, then $\|\theta - \theta - \delta\| > 0$, and then both extreme sides of (27) converges to
Lemma 11. Suppose that $H[1]-H[2]$ holds. Take $\gamma_0 > 0$ as in $H[2]$ and such that $4\gamma_0 L \leq 1$. Then for all $\gamma \in (0, \gamma_0]$, $\tilde{\Pi}_{n,\gamma}(\cdot|x_n)$ is a well-defined probability measure on $\Theta$.

Proof. Since $\gamma \mapsto h_n(\theta|\delta, \phi)$ is nondecreasing, it is enough to show that $\tilde{\Pi}_{n,\gamma}$ is well-defined for some $\gamma > 0$. We define

$$C_{n,\gamma}(\delta, \phi) \overset{\text{def}}{=} \int_{\Theta} e^{-h_n(\theta|\delta, \phi)} \mu_d(d\theta),$$

and $C_{n,\gamma} = \sum_{\delta} \pi_{\delta}(2\pi\gamma)^{\frac{d-\|\delta\|_1}{2}} \int_{\Xi} C_{n,\gamma}(\delta, \phi) G(d\phi)$. The term $C_{n,\gamma}$ is the normalizing constant of $\tilde{\Pi}_{n,\gamma}$, hence it suffices to show that $C_{n,\gamma} < \infty$ for some $\gamma > 0$ small enough.

We define $c_{\gamma} \overset{\text{def}}{=} \frac{1}{\gamma} - L - \gamma L^2$. Let us choose $\gamma_0$ such that $H[2]$ holds and $4\gamma_0 L \leq 1$. With this choice of $\gamma_0$, $c_{\gamma_0} > 0$. Set $r(\theta|\delta, \phi) \overset{\text{def}}{=} \|\nabla \ell_n(\theta|\phi) + g_n(\theta|\delta, \phi)\|^2$. For $\gamma = \gamma_0$, and using the second inequality of (27), we can bound the normalizing constant $C_{n,\gamma}$ as

$$C_{n,\gamma} = \sum_{\delta} \pi_{\delta}(2\pi\gamma)^{\frac{d-\|\delta\|_1}{2}} \int_{\Xi} \left[ \int_{\Theta} \exp \left( -h_n(\theta|\delta, \phi) \right) \mu_d(d\theta) \right] G(d\phi)$$

$$\leq \sum_{\delta} \pi_{\delta}(2\pi\gamma)^{\frac{d-\|\delta\|_1}{2}} \int_{\Xi} \left[ \int_{\Theta} \exp \left( -c_{\gamma} \sum_{j: \delta_j=0} \theta_j^2 \right) e^{\gamma r(\theta|\delta, \phi)} e^{-h_n(\theta|\delta, \phi)} \mu_d(d\theta) \right] G(d\phi)$$

$$= \sum_{\delta} \pi_{\delta}(2\pi\gamma)^{\frac{d-\|\delta\|_1}{2}} \left( \frac{\pi}{c_{\gamma}} \right)^{\frac{d-\|\delta\|_1}{2}} \int_{\Xi} \left[ \int_{\Theta} e^{\gamma r(\theta|\delta, \phi)} e^{-h_n(\theta|\delta, \phi)} \mu_{d,\delta}(d\theta) \right] G(d\phi), \quad (30)$$

and the right-hand side is finite as assumed in $H[2]$. \qed

Lemma 11. Suppose that $H[1]-H[2]$ holds and $d \geq 2$. Take $\gamma_0 > 0$ as in $H[2]$ and such that $4\gamma_0 L \leq 1$. Then for all $\gamma \in (0, \gamma_0]$,

$$\|\tilde{\Pi}_{n,\gamma} - \tilde{\Pi}_{n,\gamma}\|_{TV} \leq 2\gamma \left( \frac{5}{2} L\delta \right).$$

Proof. Recall the definitions of $C_n(\delta, \phi)$ and $C_n$ from (26) and (25), respectively. Recall also from the discussion around the definition of $\tilde{\Pi}_{n,\gamma}$ in (24) that $\tilde{\Pi}_{n,\gamma}$ can be written as

$$\tilde{\Pi}_{n,\gamma}(\delta, \phi, \theta|x_n) = \frac{1}{C_n} \pi_{\delta} \left( \frac{1}{2\pi\gamma} \right)^{\frac{d-\|\delta\|_1}{2}} \exp \left( -\frac{1}{2\gamma} \sum_{j: \delta_j=0} \theta_j^2 \right) e^{-h_n(\theta|\delta, \phi)} G(d\phi) \mu_d(d\theta).$$
With the definition of $C_{n, \gamma}$ and $C_{n, \gamma}(\delta, \phi)$ in Lemma 10 we have
\[
\tilde{\Pi}_{n, \gamma}(\delta, d\phi, d\theta|x_n) = \frac{1}{C_{n, \gamma}} \pi_\delta \left( 2\pi\gamma \right)^{\frac{|d|}{2}} e^{-h_{n, \gamma}(\theta|\delta, \phi)} G(d\phi) \mu_d(d\theta).
\]
Using the first inequality of (27), we deduce that
\[
\tilde{\Pi}_{n, \gamma}(\delta, d\phi, d\theta|x_n) \geq \frac{1}{C_{n, \gamma}} \pi_\delta \left( 2\pi\gamma \right)^{\frac{|d|}{2}} \exp \left( -\frac{1}{2\gamma} \sum_{j; \delta_j=0} \theta_j^2 \right) e^{-h_n(\theta-\delta, \phi)} G(d\phi) \mu_d(d\theta)
\]
\[
= \frac{(2\pi\gamma)^{\frac{d}{2}} C_n}{C_{n, \gamma}} \tilde{\Pi}_{n, \gamma}(\delta, d\phi, d\theta|x_n).
\]
By a standard coupling argument (see e.g. Lindvall (1992) Equation 5.1), the minorization (31) implies that
\[
\|\tilde{\Pi}_{n, \gamma} - \tilde{\Pi}_{n, \gamma}\|_{tv} \leq 2 \left( 1 - \frac{(2\pi\gamma)^{\frac{d}{2}} C_n}{C_{n, \gamma}} \right).
\]
To further bound the right-hand side of (32), we use (30) to write
\[
1 \geq \frac{(2\pi\gamma)^{\frac{d}{2}} C_n}{C_{n, \gamma}} \geq (1 - 2\gamma L - 2\gamma^2 L^2)^{d/2} \frac{\sum_\delta \pi_\delta \int_{\Xi} G(d\phi) \int_\Theta e^{-h_n(\theta|\delta, \phi)} \mu_d(d\theta)}{\sum_\delta \pi_\delta \int_{\Xi} G(d\phi) \int_\Theta e^{\gamma r(\theta|\delta, \phi)} e^{-h_n(\theta|\delta, \phi)} \mu_d(d\theta)}.
\]
We write the integral in the denominator of the last expression as
\[
\int_{\Xi} G(d\phi) \int_\Theta e^{\gamma r(\theta|\delta, \phi)} e^{-h_n(\theta|\delta, \phi)} \mu_d(d\theta) = \int_{\Xi} C_n(\delta, \phi) G(d\phi)
\]
\[
+ \gamma \int_{\Xi} G(d\phi) \int_\Theta \gamma^{-1} \left( e^{\gamma r(\theta|\delta, \phi)} - 1 \right) e^{-h_n(\theta|\delta, \phi)} \mu_d(d\theta),
\]
\[
\leq \int_{\Xi} C_n(\delta, \phi) G(d\phi) \left( 1 + \gamma \theta_\gamma \right).
\]
We conclude that
\[
\frac{(2\pi\gamma)^{\frac{d}{2}} C_n}{C_{n, \gamma}} \geq \frac{(1 - 2\gamma L - 2\gamma^2 L^2)^{d/2}}{1 + \gamma \theta_\gamma} \geq \frac{(1 - d\gamma (L + 2\gamma L^2))}{1 + \gamma \theta_\gamma},
\]
where the last inequality uses the fact that $d \geq 2$, and $(1 - x)^a \geq 1 - ax$ for all $x \in (0, 1)$, $a \geq 1$. Now, (33) with (32) easily implies the stated bound. \hfill \Box

7.2. Proof of Theorem 5. We will show that $\tilde{\Pi}$ holds and there exists $\gamma_0 > 0$, such that (10) holds. The theorem then follows from Remark 4 and Theorem 3.

For any $\sigma^2 > 0$, the function $\ell_n(\cdot|\sigma^2)$ is clearly convex and $\nabla \ell_n(\theta|\sigma^2) = -\frac{1}{\sigma^2} X'(y - X\theta)$. Hence
\[
\|\nabla \ell_n(\theta_2|\sigma^2) - \nabla \ell_n(\theta_1|\sigma^2)\| \leq \frac{\lambda_{\max}(X'X)}{\sigma^2} \|\theta_2 - \theta_1\|, \quad \theta_1, \theta_2 \in \mathbb{R}^d.
\]
Hence \( I(1) \) holds with \( L = \lambda_{\text{max}}(X'X)/\sigma^2 \). The density \( p \) in \( (11) \) is log-concave, which implies that \( P_n(\cdot, \sigma^2, \delta, \phi) \) is convex for any \( \sigma^2 > 0 \), and any \((\delta, \phi) \in \Delta \times \Xi\). This proves \( I(1) \).

For \( \theta \in \Theta_2 \), \( \text{sign}(\theta) \) is a subgradient of \( x \mapsto \|x\|_1 \) at \( \theta \). Hence \( g_n(\theta, \delta, \phi) \) \( = \int_{\Theta_2} \frac{\alpha \lambda_1}{\sigma^2} \text{sign}(\theta) + \frac{1 - \alpha}{\sigma^2} \lambda_2 \theta \) is a subgradient of \( P_n(\cdot, \delta, \phi) \) at \( \theta \in \Theta_2 \). Hence, as explained in Remark 4, to prove \( H_2 \), it suffices to show that there exists \( \gamma > 0 \) such that for all \( \delta \in \Delta \), the integral

\[
\int_a^M d\lambda_1 \int_a^M d\lambda_2 \int_{\Theta_2} r_n(\theta, \delta, \phi) e^{\gamma r_n(\theta, \delta, \phi)} e^{-h_n(\theta, \sigma^2, \delta, \phi)} \mu_d, d(\theta)
\]

is finite, where \( r_n(\theta, \delta, \phi) \) \( = \int \|g_n(\theta, \delta, \phi)\|^2 + \|\nabla \ell_n(\theta, \sigma^2)\|^2 \). We easily bound \( r_n \) to obtain

\[
r_n(\theta, \delta, \phi) \leq \frac{2}{\sigma^2} \left( \sigma^2 + \frac{1}{\sigma^2} \right) + \frac{2}{\sigma^2} \left( (1 - \alpha)^2 \lambda_2^2 + \lambda_{\text{max}}(X'X)^2 \right) \|\theta\|^2.
\]

Furthermore,

\[
\|\nabla \ell_n(\theta, \sigma^2)\|^2 = \frac{1}{\sigma^2} \|y - X\theta\|^2 (X'X)(y - X\theta) \leq \frac{\lambda_{\text{max}}(X'X)}{\sigma^2} \ell_n(\theta, \sigma^2),
\]

and

\[
\|g_n(\theta, \delta, \phi)\|^2 \leq \frac{\alpha^2 \lambda_1^2 d}{\sigma^2} + \frac{(1 - \alpha)^2 \lambda_2^2 \|\theta\|^2}{\sigma^4} + \frac{2\alpha(1 - \alpha)\lambda_1 \lambda_2 \|\theta\|_1}{\sigma^4} + \frac{(1 - \alpha)^2 \lambda_2 \|\theta\|^2}{2\sigma^2} + \frac{\alpha \lambda_1 \|\theta\|_1}{\sigma^2}.
\]

Since \( \lambda_2 < M < \frac{\lambda_{\text{max}}(X'X)}{(1 - \alpha)} \), we have \( 2\gamma(1 - \alpha)\lambda_2 / \sigma^2 < \frac{2\gamma \lambda_{\text{max}}(X'X)}{\sigma^2} \leq \frac{2\gamma \lambda_{\text{max}}(X'X)}{\sigma^2} \leq \frac{1}{2} \). Hence, \( r_n(\theta, \delta, \phi) = \|g_n(\theta, \delta, \phi)\|^2 + \|\nabla \ell_n(\theta, \sigma^2)\|^2 \) satisfies for \( \theta \in \Theta_2 \),

\[
e^{\gamma r_n(\theta, \delta, \phi)} e^{-h_n(\theta, \sigma^2, \delta, \phi)} \leq e^{-\frac{\gamma n \lambda_2^2 d}{\sigma^2} \left( \frac{1}{Z} \right) \|\delta\|_1} \exp \left[ -\left( 1 - \frac{\gamma \lambda_{\text{max}}(X'X')}{\sigma^2} \right) \ell_n(\theta, \sigma^2) - \left( 1 - \frac{2\gamma(1 - \alpha)\lambda_2}{\sigma^2} \right) \left( \frac{1}{2} \right) \left( \frac{(1 - \alpha)\lambda_2 \|\theta\|^2}{2\sigma^2} + \frac{\alpha \lambda_1 \|\theta\|_1}{\sigma^2} \right) \right] \leq e^{-\frac{\gamma n \lambda_2^2 d}{\sigma^2} \left( \frac{1}{Z} \right) \|\delta\|_1} \exp \left[ -\frac{1}{2} \left( \frac{(1 - \alpha)\lambda_2 \|\theta\|^2}{2\sigma^2} + \frac{\alpha \lambda_1 \|\theta\|_1}{\sigma^2} \right) \right],
\]

where we recall that \( Z = \tilde{Z}_\alpha \left( \frac{\lambda_1}{\sigma^2}, \frac{\lambda_2}{\sigma^2} \right) \), with \( \tilde{Z}_\alpha \) as defined in (12). This, and (36) easily implies that the integral (35) is finite.
7.3. Proof of Theorem 7. The proof is very similar to the proof of Theorem 5. For $\theta, H \in \mathbb{R}^d$, 

$$\langle \nabla \ell_n(\theta | \sigma^2), H \rangle = \text{Tr} \left( (L(H)'y' (L(\theta)\text{diag}(\sigma^2) + I_p) \right).$$

Hence, for $\theta_1, \theta_2 \in \mathbb{R}^d$, and $H \in \mathbb{R}^d$ we have

$$\left| \langle \nabla \ell_n(\theta_1 | \sigma^2) - \nabla \ell_n(\theta_1 | \sigma^2), H \rangle \right| = \left| \text{Tr} \left( (L(H)'y' yL(\theta_1 - \theta_2)\text{diag}(\sigma^2) \right) \right|$$

$$= \left| \sum_{j=1}^{p} \sigma_j^2 \langle L(H)_j, y'yL(\theta_1 - \theta_2)_j \rangle \right|$$

$$\leq \sigma^2 \lambda_{\text{max}}(y' y) \sum_{j=1}^{p} \|L(H)_j\| \|L(\theta_1 - \theta_2)_j\|$$

$$\leq 2\sigma^2 \lambda_{\text{max}}(y' y)\|H\| \|\theta_1 - \theta_2\|,$$

which implies that

$$\|\nabla \ell_n(\theta_1 | \sigma^2) - \nabla \ell_n(\theta_1 | \sigma^2)\| \leq 2\sigma^2 \lambda_{\text{max}}(y' y)\|\theta_1 - \theta_2\|.$$

We conclude that $\ell_n$ has a Lipschitz gradient with Lipschitz constant $L = 2\sigma^2 \lambda_{\text{max}}(y' y)$, and $H \Box$ holds.

We can also write

$$\|\nabla \ell_n(\theta | \sigma^2)\|^2 = \sup_{\|H\|=1} \langle \nabla \ell_n(\theta | \sigma^2), H \rangle^2 = \sup_{\|H\|=1} \text{Tr} \left( (L(H)'y' y(L(\theta)\text{diag}(\sigma^2) + I_p) \right)^2,$$

and

$$\text{Tr} \left( (L(H)'y' y(L(\theta)\text{diag}(\sigma^2) + I_p) \right) = \sum_{j=1}^{p} \sigma_j^2 \left\langle yL(H)_j, y \left( L(\theta) + \text{diag} \left( \frac{1}{\sigma^2} \right) \right)_j \right\rangle$$

$$\leq \lambda_{\text{max}}(y' y)^{1/2} \sum_{j=1}^{p} \sigma_j^2 \|L(H)_j\| \left\| y \left( L(\theta) + \text{diag} \left( \frac{1}{\sigma^2} \right) \right)_j \right\|$$

$$\leq 2\sigma \lambda_{\text{max}}(y' y)^{1/2}\|H\| \left\{ \sum_{j=1}^{p} \sigma_j^2 \left\| y \left( L(\theta) + \text{diag} \left( \frac{1}{\sigma^2} \right) \right)_j \right\|^2 \right\}^{1/2}.$$

It follows that

$$\|\nabla \ell_n(\theta | \sigma^2)\|^2 \leq 4\ell_n(\theta | \sigma^2).$$

With this last property, the rest of the proof is similar to the proof of Theorem 5.
References


