Homework 7 (Math/Stats 425, Winter 2013)

Due Tuesday April 9, in class

1. The annual rainfall (in inches) in a certain region is modeled as being normally distributed with $\mu = 40$ and $\sigma = 4$. According to this model, what is the probability that it will take over 10 years before a year occurs having rainfall above 50 inches? What assumptions are you making?

Solution: Let $X$ denote the annual rainfall, and $E$ denote the event that it will take over 10 years starting from this year before a year occurs having a rainfall of over 50 inches. Then

$$P(X > 50) = 1 - P(X \leq 50)$$

$$= 1 - \int_{-\infty}^{50} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \, dx$$

$$= 1 - \int_{-\infty}^{2.5} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \, du$$

$$= 1 - \Phi(2.5)$$

$$= 1 - 0.9938.$$

Therefore,

$$P(E) = \left(1 - (1 - 0.9938)\right)^{10} = 0.9397$$

We are assuming that the annual rainfall is independent from year to year.

2. The number of years a new radio functions is exponentially distributed with parameter $\lambda = 1/8$. If Jones buys a used radio, what is the probability that it will be working after an additional 8 years? Comment on your assumptions.

Solution:

$$P(X > 8 + x | X > x) = P(X > 8)$$

$$= \int_{-\infty}^{\infty} \lambda e^{-\lambda x} \, dx$$

$$= e^{-1}.$$

3. Find the density function of $R = a \sin(\Theta)$, where $a$ is a fixed constant and $\Theta$ is uniformly distributed on $(-\pi/2, \pi/2)$.

Note: such a random variable $R$ arises in the theory of ballistics. If a projectile is fired from the origin at an angle $\alpha$ from the earth with speed $\nu$, then the point $R$ at which it returns to the earth can be expressed as $R = (\nu^2/g) \sin(2\alpha)$ where $g$ is the gravitational constant.
Solution: First, calculate cdf of $R$.

$$F_R(r) = P\{Asin\Theta \leq r\} = P\{sin\Theta \leq \frac{r}{A}\} = P\{\Theta \leq arcsin\left(\frac{r}{A}\right)\} = \frac{arcsin\left(\frac{r}{A}\right) + \frac{\pi}{2}}{\frac{\pi}{2} + \frac{\pi}{2}} = \frac{1}{\pi} arcsin\left(\frac{r}{A}\right) + \frac{1}{2}$$

By differentiating cdf, we get pdf,

$$f_R(r) = \frac{d}{dr} \left(\frac{1}{\pi} arcsin\left(\frac{r}{A}\right) + \frac{1}{2}\right) = \frac{1}{\pi \sqrt{1 - \left(\frac{r}{A}\right)^2}} \quad \text{for} \quad -A \leq r \leq A$$

4. Suppose that 3 balls are chosen successively, without replacement, from an urn containing 5 white balls and 8 red balls. Let $X_i$ equal 1 if the $i$th ball drawn is white, and otherwise $X_i$ equals 0. Write the joint probability mass function of

(a) $X_1$ and $X_2$
(b) $X_1$, $X_2$ and $X_3$
(c) $X_1 + X_2$ and $X_1 + X_3$

Solution:

(a)

\[
p(0, 0) = \frac{8}{13} \cdot \frac{7}{12} = \frac{14}{39} \quad p(0, 1) = \frac{8}{13} \cdot \frac{5}{12} = \frac{10}{39} \\
p(1, 0) = \frac{5}{13} \cdot \frac{8}{12} = \frac{10}{39} \quad p(1, 1) = \frac{5}{13} \cdot \frac{4}{12} = \frac{5}{39}
\]

(b)

\[
p(0, 0, 0) = \frac{8}{13} \cdot \frac{7}{12} \cdot \frac{6}{11} = \frac{28}{143} \quad p(0, 0, 1) = \frac{8}{13} \cdot \frac{7}{12} \cdot \frac{5}{11} = \frac{70}{429}
\]
Similarly,
\[
\begin{align*}
p(0,1,0) &= p(1,0,0) = \frac{70}{429} \\
p(0,1,1) &= \frac{854}{131211} = \frac{40}{429}
\end{align*}
\]
Similarly,
\[
\begin{align*}
p(1,0,1) &= p(1,1,0) = \frac{40}{429} \\
p(1,1,1) &= \frac{543}{131211} = \frac{5}{143}
\end{align*}
\]
(c)
\[
\begin{align*}
p(0,0) &= \frac{28}{143} \\
p(0,1) &= \frac{70}{429} \\
p(0,2) &= 0 \\
p(1,0) &= \frac{70}{429} \\
p(1,1) &= \frac{40}{429} + \frac{70}{429} \\
p(1,2) &= \frac{40}{429} \\
p(2,0) &= 0 \\
p(2,1) &= \frac{40}{429} \\
p(2,2) &= \frac{543}{131211} = \frac{5}{143}
\end{align*}
\]
5. The joint probability density function of \(X\) and \(Y\) is given by
\[
f(x, y) = \frac{6}{7} \left( x^2 + \frac{xy}{2} \right) \quad 0 < x < 1, \ 0 < y < 2
\]
(a) Verify that this is indeed a valid joint density function.
(b) Compute the marginal density function of \(X\).
(c) Find \(\mathbb{P}(X > Y)\).
(d) Find \(\mathbb{P}(Y > 1/2 \mid X < 1/2)\).
(e) Find \(\mathbb{E}[X]\).
(f) Find \(\mathbb{E}[Y]\)
Solution:
(a) \[
\int_0^1 \int_0^2 \frac{6}{7} (x^2 + \frac{xy}{2}) dydx = \int_0^1 \int_0^2 \frac{6}{7} (2x^2 + x) dx = 1
\]
(b) \[ f_X(x) = \int_0^2 \frac{6}{7} (x^2 + \frac{xy}{2}) \, dy = \frac{6}{7} (2x^2 + x) \]

(c) \[
P(X > Y) = \int_0^1 \int_0^x \frac{6}{7} (x^2 + \frac{xy}{2}) \, dy \, dx \\
= \frac{6}{7} \int_0^1 (x^2 + \frac{1}{4}xy^2) \bigg|_0^x \, dx \\
= \frac{6}{7} \int_0^1 \frac{5}{4}x^3 \, dx = \frac{15}{56}
\]

(d) \[
P(X < \frac{1}{2}) = \int_0^{\frac{1}{2}} f_X(x) \, dx = \int_0^{\frac{1}{2}} \frac{6}{7} (2x^2 + x) \, dx = \frac{6}{7} \left( \frac{2x^3}{3} + \frac{x^2}{2} \right) \bigg|_0^{\frac{1}{2}} = \frac{5}{28}
\]

\[
P(X < \frac{1}{2}, Y > \frac{1}{2}) = \int_0^{\frac{1}{2}} \int_{\frac{1}{2}}^1 \frac{6}{7} (x^2 + \frac{xy}{2}) \, dx \, dy = \frac{3}{56} + \frac{45}{28 \times 16}
\]

By the definition of conditional probability,
\[
P(Y > \frac{1}{2} \mid X < \frac{1}{2}) = \frac{P(X < \frac{1}{2}, Y > \frac{1}{2})}{P(X > \frac{1}{2})} = \frac{3}{56} \times \frac{28}{5} + \frac{48}{28 \times 16} \times \frac{5}{28} = \frac{69}{80} = 0.8625
\]

(e) \[
EX = \int_0^1 x f_X(x) \, dx \\
= \int_0^1 x \left( \frac{12}{7} x^2 + \frac{6}{7} x \right) \, dx \\
= \left[ \frac{12}{7} \frac{x^4}{4} + \frac{6}{7} \frac{x^3}{3} \right]_0^1 \\
= \frac{5}{7}
\]

(f) \[
EY = \int_0^2 \int_0^1 y \frac{6}{7} (x^2 + \frac{xy}{2}) \, dx \, dy \\
= \frac{6}{7} \int_0^1 (2x^2 + \frac{4}{3} x) \, dx \\
= \left[ \frac{6}{7} \left( \frac{2}{3} x^3 + \frac{2}{3} x^2 \right) \right]_0^1 \\
= \frac{8}{7}
\]
Recommended reading:
Sections 5.4, 5.5, 5.7 and 6.1 in Ross “A First Course in Probability,” 8th edition. This course will not include the material in Section 5.6.