3. Conditional probability & independence

Conditional Probabilities

- **Question**: How should we modify $\mathbb{P}(E)$ if we learn that event $F$ has occurred?

- **Derivation**: Suppose we repeat the experiment $n$ times. Let $n(E \cap F)$ be the number of times that both $E$ and $F$ occur, and $n(F)$ the number of times $F$ occurs.

- The proportion of times $E$ occurs only counting trials where $F$ occurs is

\[
\frac{n(E \cap F)}{n(F)} = \frac{n(E \cap F)/n}{n(F)/n} \approx \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}.
\]

- **Definition**: the conditional probability of $E$ given $F$ is

\[
\mathbb{P}(E | F) = \frac{\mathbb{P}(E \cap F)}{\mathbb{P}(F)}, \quad \text{for} \quad \mathbb{P}(F) > 0
\]
Example 1. 27 students out of a class of 43 are engineers. 20 of the students are female, of whom 7 are engineers. Find the probability that a randomly selected student is an engineer given that she is female.
Example 2. Deal a 5 card poker hand, and let

\( E = \{ \text{at least 2 aces} \}, \quad F = \{ \text{at least 1 ace} \}, \)

\( G = \{ \text{hand contains ace of spades} \}. \)

(a) Find \( \mathbb{P}(E) \)

(b) Find \( \mathbb{P}(E \mid F) \)

(c) Find \( \mathbb{P}(E \mid G) \)
The Multiplication Rule

- Re-arranging the conditional probability formula gives

\[ \Pr(E \cap F) = \Pr(F) \Pr(E | F) \]

This is often useful in computing the probability of the intersection of events.

Example. Draw 2 balls at random without replacement from an urn with 8 red balls and 4 white balls. Find the chance that both are red.
The General Multiplication Rule

\[
P(E_1 \cap E_2 \cap \cdots \cap E_n) = \\
P(E_1) \times P(E_2 | E_1) \times P(E_3 | E_1 \cap E_2) \times \\
\quad \cdots \times P(E_n | E_1 \cap E_2 \cap \cdots \cap E_{n-1})
\]

Example 1. Anil and Beth roll two dice, and play a game as follows. If the total is 5, A wins. If the total is 7, B wins. Otherwise, they play a second round, and so on. Find \( P(E_n) \), for \( E_n = \{ \text{A wins on nth round} \} \).
Example 2. I have $n$ keys, one of which opens a lock. Trying keys at random without replacement, find the chance that the $k$th try opens the lock.
The Law of Total Probability

• From axiom A3, \( \mathbb{P}(E) = \mathbb{P}(E \cap F) + \mathbb{P}(E \cap F^c) \).

Using the definition of conditional probability,

\[
\mathbb{P}(E) = \mathbb{P}(E | F) \mathbb{P}(F) + \mathbb{P}(E | F^c) \mathbb{P}(F^c)
\]

• This is extremely useful. It may be difficult to compute \( \mathbb{P}(E) \) directly, but easy to compute it once we know whether or not \( F \) has occurred.

• To generalize, say events \( F_1, \ldots, F_n \) form a partition if they are disjoint and \( \bigcup_{i=1}^{n} F_i = S \).

• Use a Venn diagram to argue that

\[
\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E \cap F_i).
\]

• Apply conditional probability to give the law of total probability,

\[
\mathbb{P}(E) = \sum_{i=1}^{n} \mathbb{P}(E | F_i) \mathbb{P}(F_i)
\]
Example 1. Eric’s girlfriend comes round on a given evening with probability 0.4. If she does not come round, the chance Eric watches *The Wire* is 0.8. If she does, this chance drops to 0.3. Find the probability that Eric gets to watch *The Wire*. 
Bayes Formula

- Sometimes $\mathbb{P}(E | F)$ may be specified and we would like to find $\mathbb{P}(F | E)$.

Example 2. I call Eric and he says he is watching *The Wire*. What is the chance his girlfriend is around?

- A simple manipulation gives **Bayes’ formula**,

$$
\mathbb{P}(F | E) = \frac{\mathbb{P}(E | F) \mathbb{P}(F)}{\mathbb{P}(E)}
$$

Proof.

- Combining this with the law of total probability,

$$
\mathbb{P}(F | E) = \frac{\mathbb{P}(E | F) \mathbb{P}(F)}{\mathbb{P}(E | F) \mathbb{P}(F) + \mathbb{P}(E | F^c) \mathbb{P}(F^c)}
$$
Solution to Example 2.

- this computation can be viewed using a tree:
Sometimes conditional probability calculations can give quite unintuitive results.

**Example 3.** I have three cards. One is red on both sides, another is red on one side and black on the other, the third is black on both sides. I shuffle the cards and put one on the table, so you can see that the upper side is red. What is the chance that the other side is black?

• is it $1/2$, or $>1/2$ or $<1/2$?

**Solution**
Discussion problem. Suppose 99% of people with HIV test positive, 95% of people without HIV test negative, and 0.1% of people have HIV. What is the chance that someone testing positive has HIV?
Example: Statistical inference via Bayes’ formula

Rosencrantz and Guildenstern play a game where R tosses a coin, and wins $1 if it lands on H or loses $1 on T. G is surprised to find that he loses the first ten times they play. If G’s prior belief is that the chance of R having a two headed coin is 0.01, what is his posterior belief?

Note. Prior and posterior beliefs are assessments of probability before and after seeing an outcome. The outcome is called data.

Solution.
Independence

• Heuristically, \( E \) is independent of \( F \) if the chance of \( E \) occurring is not affected by whether \( F \) occurs, i.e.,

\[
P(E \mid F) = P(E)
\]  

(1)

• We say that \( E \) and \( F \) are **independent** if

\[
P(E \cap F) = P(E)P(F)
\]  

(2)

**Note.** (2) is a rearrangement of (1). Check this!

**Note.** It is clear from (2) that independence is a symmetric relationship. Also, (2) is properly defined when \( P(F) = 0 \).

**Note.** (1) is a useful way to think about independence; (2) is usually better to do the math.
Proposition. If $E$ and $F$ are independent, then so are $E$ and $F^c$.

Proof.
Example 1: Independence can be obvious
Draw a card from a shuffled deck of 52 cards. Let $E$ = card is a spade and $F$ = card is an ace. Are $E$ and $F$ independent?

Solution

Example 2: Independence can be surprising
Toss a coin 3 times. Define $A$ = {at most one T} = \{HHH, HHT, HTH, THH\}
$B$ = {both H and T occur} = \{HHH, TTT\}^c.
Are $A$ and $B$ independent?

Solution
Independence as an Assumption

• It is often convenient to suppose independence. People sometimes assume it without noticing.

Example. A sky diver has two chutes. Let

\[ E = \{ \text{main chute opens} \}, \quad P(E) = 0.98; \]
\[ F = \{ \text{backup opens} \}, \quad P(F) = 0.90. \]

Find the chance that at least one opens, making any necessary assumption clear.

Note. Assuming independence does not justify the assumption! Both chutes could fail because of the same rare event, such as freezing rain.
Independence of Several Events

- Three events $E, F, G$ are **independent** if
  \[
  \Pr(E \cap F) = \Pr(E) \cdot \Pr(F)
  \]
  \[
  \Pr(F \cap G) = \Pr(F) \cdot \Pr(G)
  \]
  \[
  \Pr(E \cap G) = \Pr(E) \cdot \Pr(G)
  \]
  \[
  \Pr(E \cap F \cap G) = \Pr(E) \cdot \Pr(F) \cdot \Pr(G)
  \]

- If $E, F, G$ are independent, then $E$ will be independent of any event formed from $F$ and $G$.

**Example.** Show that $E$ is independent of $F \cup G$.

**Proof.**
Pairwise Independence

- $E$, $F$ and $G$ are **pairwise independent** if $E$ is independent of $F$, $F$ is independent of $G$, and $E$ is independent of $G$.

**Example.** Toss a coin twice. Set $E = \{HH, HT\}$, $F = \{TH, HH\}$ and $G = \{HH, TT\}$.

(a) Show that $E$, $F$ and $G$ are pairwise independent.

(b) By considering $\mathbb{P}(E \cap F \cap G)$, show that $E$, $F$ and $G$ are NOT independent.

**Note.** Another way to see the dependence is that $\mathbb{P}(E \mid F \cap G) = 1 \neq \mathbb{P}(E)$. 
Example: Independent trials
A sequence of $n$ independent trials results in a success with probability $p$ and a failure with probability $1 - p$. What is the probability that

• at least one success occurs?

• exactly $k$ successes occur?
**Gambler’s Ruin Problem.** A and B play an independent sequence of games. Each game, the winner gets one dollar from the loser, and play continues until one player is bankrupt. A starts with \( i \) dollars and B starts with \( N - i \) dollars. A wins each game with probability \( p \). What is the probability that A ends up with all the money?
Conditional probability obeys the axioms

Let $Q_F(E) = \mathbb{P}(E \mid F)$. Then

- $0 \leq Q_F(E) \leq 1$
- $Q_F(\mathbb{S}) = 1$
- If $E_1, E_2, \ldots$ are disjoint, then

$$Q_F(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} Q_F(E_i)$$

Since $Q_F$ obeys the axioms, all our previous propositions for probabilities give analogous results for conditional probabilities.

**Examples**

$$\mathbb{P}(E^c \mid F') = 1 - \mathbb{P}(E \mid F')$$

$$\mathbb{P}(A \cup B \mid F) = \mathbb{P}(A \mid F) + \mathbb{P}(B \mid F) - \mathbb{P}(A \cap B \mid F)$$
Example: Insurance policies re-visited
Insurance companies categorize people into two groups: accident prone (30%) or not. An accident prone person will have an accident within one year with probability 0.4; otherwise, 0.2. What is the conditional probability that a new policyholder will have an accident in his second year, given that the policyholder has had an accident in the first year?