8. Overview of further topics

The Weak Law of Large Numbers

• If $X_1, X_2, \ldots$ are i.i.d. RVs, with common mean $\mathbb{E}(X_i) = \mu$, then for any $\epsilon > 0$

$$\mathbb{P}\left(\left|\frac{X_1 + \cdots + X_n}{n} - \mu\right| > \epsilon\right) \rightarrow 0$$

in the limit as $n \rightarrow \infty$. This is the weak law of large numbers (WLLN).

• The WLLN says that the sample average (for an i.i.d. sample of size $n$) converges to the expectation.

• This theorem can also be described as saying that the sample mean converges to the population mean.

• The WLLN says nothing about the rate of convergence. It also applies if $\text{Var}(X) = \infty$. 
Chebyshev’s Inequality

• To prove the WLLN, we first obtain Chebyshev’s inequality.

Proposition. If $X$ is any RV, with $\mathbb{E}(X) = \mu$, $\text{Var}(X) = \sigma^2$, then

$$\mathbb{P}(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \text{ for any } \epsilon > 0$$

Proof.
Example. Suppose the daily change in value of stock is an i.i.d. sequence $X_1, X_2, \ldots$ with $\mathbb{E}(X_i) = 0$ and $\text{Var}(X_i) = 1$. What can you say, using Chebyshev’s inequality, about the chance that the value changes by more than 5 in 10 days?

Solution.
Proof of Weak Law of Large Numbers

- $X_1, X_2, \ldots$ are i.i.d. with mean $\mu$.
- Suppose an additional, unnecessary, condition that $X_1, X_2, \ldots$ have variance $\sigma^2$.
- Apply Chebyshev’s inequality to $\bar{X}_n - \mu$ for
  \[
  \bar{X}_n = \frac{X_1 + \cdots + X_n}{n}
  \]
The Central Limit Theorem

- If $X_1, X_2, \ldots$ are i.i.d., with mean $\mu$ and variance $\sigma^2$, then for any constant $a$,

$$
P \left( \frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}} \leq a \right) \to P(Z \leq a)
$$

in the limit as $n \to \infty$, where $Z$ is standard normal, i.e. $Z \sim N(0, 1)$.

Comments on the CLT

- The CLT may be re-written as

$$
\frac{X_1 + \cdots + X_n - n\mu}{\sigma \sqrt{n}} \to Z
$$

where the limit is interpreted as convergence of the c.d.f. (this type of limit is called convergence in distribution). This in turn can be rewritten as

$$
\frac{\sqrt{n}(\bar{X}_n - \mu)}{\sigma} \to Z.
$$

- A remarkable thing about the CLT is that the behavior of the average depends only on the mean and the variance.
More Comments

• The CLT can be proved, but we can also view it as an empirical result. The CLT proposes a normal approximation for the distribution of an average; an approximation which can be tested by a computer experiment. How?

• The CLT “often” gives a good approximation for $n$ as small as 10 or 20.

• The closer $X_1, X_2, \ldots$ are to having the normal distribution, the smaller the $n$ required for a good approximation.

• If $X_1, X_2, \ldots$ are themselves i.i.d. $N(\mu, \sigma^2)$ then it is exactly true that $\frac{\sqrt{n}}{\sigma}(\bar{X}_n - \mu)$ has the standard normal distribution.
Example. A die is rolled 10 times. Use the CLT to approximate the chance that the sum is between 25 and 45.

Solution.
Example. A sequence of independent trials is carried out, each with chance $p$ of success. Let $M$ be the number of failures preceding the first success, and $N$ the number of failures between the first two successes. Find the joint probability mass function of $M$ and $N$.

Solution.
Example. A die is rolled repeatedly. Find the probability that the first roll is strictly greater than the next $k$ rolls (i.e. if the values of the rolls are $X_1, X_2, \ldots$ then $X_1 > X_j$ for $j = 2, \ldots, k + 1$).

Solution.
Example. A die is thrown \( N \) times. Let \( X \) be the number of times the die lands showing six spots, and \( Y \) the number of times it lands showing five spots. Find the mean and variance of \( Z = X - Y \).

Solution.
Example. If $X$ and $Y$ are independent and identically distributed Uniform$[0, 1]$ random variables, find the density of $Z = X/(X + Y)$.

Solution.
Example. Suppose $X$ and $Y$ have joint density

$$f(x, y) = ce^{-(x+y^2)}$$

on the region $x \geq 0$ and $-\infty < y < \infty$, with $c$ being an unknown constant. Find the expected value of $X + Y^2$. You do not necessarily have to do any integration!

Solution.